Chapter 10

Miscellaneous topics - 2

Lecture 36

Topics

10.4.4 Response to disturbance of two degrees of freedom system with complex pair of roots

10.4.5 Response of general aviation airplane (Navion) to sudden application of elevator deflection

10.5 Transfer functions for airplane motion

10.6 Concluding remark

10.4.4 Response to disturbance of two degrees of freedom system with complex pair of roots

Consider the following modified form of Eqs.(10.17) and (10.18) :

$$3\dot{x}_1 + 2x_1 - \dot{x}_2 = 0, \tag{10.45}$$

$$\dot{x}_1 + 4\dot{x}_2 + 3x_2 = 0 \tag{10.46}$$

The initial conditions to obtain the response to a disturbance are taken as :

$x_1(0) = 1$, $x_2(0) = 0$.

Substituting $x_1 = \rho_1 e^{\lambda t}$ and $x_2 = \rho_2 e^{\lambda t}$ gives:

$$3\rho_1 \lambda + 2\rho_1 - \rho_2\lambda = 0 \tag{10.47}$$

$$\rho_1\lambda + 4\rho_2 \lambda + 3\rho_2 = 0 \tag{10.48}$$

A non-trivial solution of Eqs.(10.47) and (10.48) requires :

$$\begin{vmatrix} 3\lambda + 2 & -\lambda \\ \lambda & 4\lambda + 3 \end{vmatrix} = 0$$

$$\text{Or } 13\lambda^2 + 17\lambda + 6 = 0 \tag{10.49}$$

Hence,

$$\lambda_{1,2} = \frac{-17 \pm \sqrt{17^2 - 4\times13\times6}}{2\times13} = \frac{-17 \pm i\sqrt{23}}{26} \tag{10.50}$$

$$= -0.654 \pm i\,0.185$$

For convenience in subsequent calculations $\lambda_{1,2}$ is expressed as -r ± i s.

Since, the roots are complex, the response can be written as :

Let, $x_1 = C_1 e^{-rt} \cos st + C_2 e^{-rt} \sin st$ (10.51)

$x_2 = D_1 e^{-rt} \cos st + D_2 e^{-rt} \sin st$ (10.52)

To evaluate C_1,C_2 , D_1 and D_2, substitute x_1 and x_2 in Eq.(10.45). This gives:

$3C_1 [-re^{-rt} \cos st – s e^{-rt} \sin st] + 3C_2[- re^{-rt} \sin st + s e^{-st} \cos st]$

$+ 2 [C_1 e^{-rt} \cos st + C_2 e^{-st} \sin t]$

$– D_1 [-re^{–rt} \cos st –s e^{–rt} \sin st] – D_2 [-r e^{–rt} \sin st+ s e^{–rt} \cos st] = 0.$ (10.53)

Canceling $e^{–rt}$ and collecting coefficients of cos st and sin st gives:

$[-3r C_1 + 3s C_2 +2C_1 + r D_1 – s D_2] \cos st +$

$[-3s C_1- 3r C_2 + 2C_2 +s D_1 +r D_2] \sin st = 0.$ (10.54)

Since, sine and cosine are linearly independent functions, each term in Eq.(10.54) must be zero.

Or $(2 - 3r)C_1 + 3s C_2 + r D_1 – s D_2 = 0$ (10.55)

and $- 3 s C_1 + (2 - 3r)C_2 +s D_1 + r D_2 = 0.$ (10.56)

Equations (10.57) and (10.58) provide two equations for C_1, D_1, C_2 and D_2. Two additional equations are obtained by satisfying initial considerations. At t = 0, x_1 equals one and x_2 equals zero. Substituting these in Eqs.(10.51) and (10.52) gives

$1 = C_1+0$ or $C_1 = 1$ (10.57)

$0 = D_1+ 0$ or $D_1 = 0$ (10.58)

Substituting C_1 and D_1 in Eqs.(10.55) and (10.56) gives:

$(2-3r) + 3s C_2 – s D_2 = 0$ (10.59)

$- 3s + (2 - 3r) C_2 + r D_2 = 0$ (10.60)

Substituting r = 0.654 and s = 0.185 gives:

$(2 – 3 \times 0.654) + 3 \times 0.185 C_2 – 0.185 D_2 = 0$

Or $0.038 + 0.555 C_2 - 0.185 D_2 = 0$ (10.61)

and $-0.555 + 0.038 C_2 + 0.654 D_2 = 0$ (10.62)

Solving Eqs.(10.61) and (10.62) gives $C_2 = 0.2104$, $D_2 = 0.8365$.

Hence, the response is:

$x_1 = e^{-0.654 t} \cos 0.185 t + 0.2104 e^{-0.654t} \sin 0.185t$ (10.63)

$x_2 = 0.8365\ e^{-0.654\ t} \sin 0.185\ t$ (10.64)

Remarks:

i) The same result as above would be obtained if Eq.(10.46) is used to evaluate C_2 and D_2.

ii) The variations of x_1 and x_2 with time are plotted in Figs.10.6a and b. Note $t_{1/2} = 0.693 / 0.654 = 1.06$ s and Time period = $2\pi / 0.185 = 33.95$ s. Since, the damping is heavy and the time period is long, the negative part of the curves are not seen clearly.

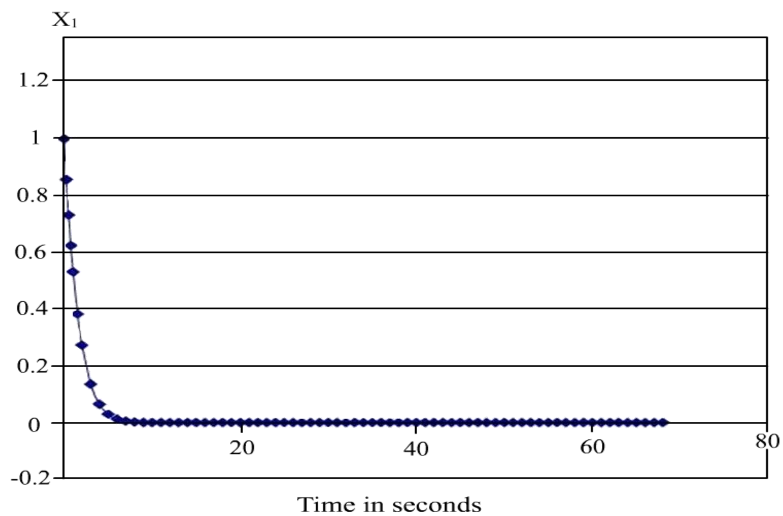

Fig.10.6a Response of a two degrees of freedom system with roots as complex pair but with heavy damping -variation of x_1 with time

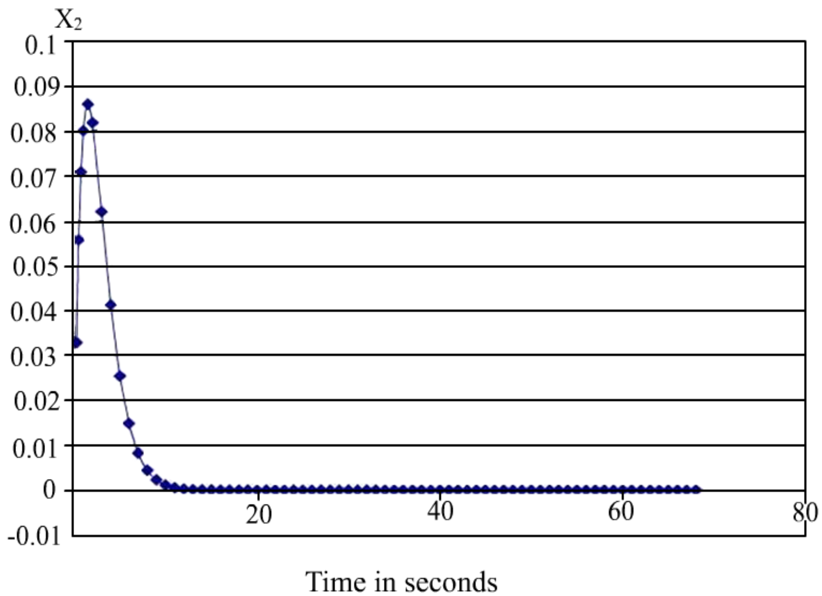

Fig.10.6b Response of a two degrees of freedom system with roots as complex pair but with heavy damping -variation of x_2 with time

10.4.5 Response of general aviation airplane (Navion) to sudden application of elevator deflection:

The calculation of the response of an airplane to a disturbance or to control surface deflection, is more involved but conceptually not difficult. Figures 8.3a, 8.3b, 8.4a and 8.4b show the response of the general aviation airplane (Navion) to disturbance in the longitudinal case. Figures 9.3a to 9.3d show the response to disturbance in the lateral case.

Response to sudden application of one degree elevator deflection, taken from Ref. 1.12, chapter 6, is shown in Figs.10.7a,b and c. It is observed that the change in angle of attack takes place within the first few seconds after the deflection of the elevator. This is due to the rapid response to the short period mode. However, approach to the final values of $\Delta\alpha$ and Δu is gradual due to slow response of phugoid.

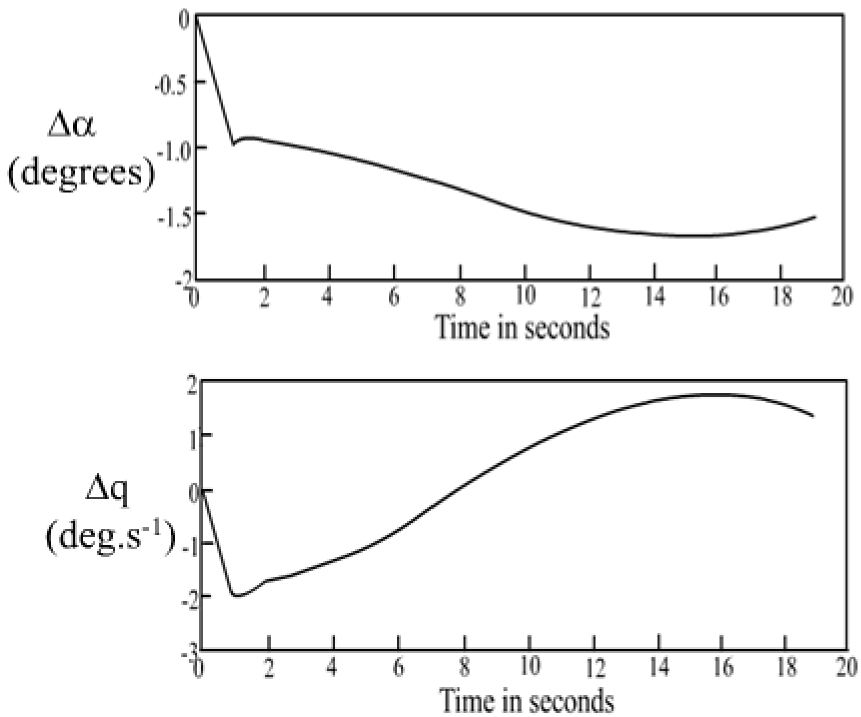

Fig.10.7a Response of the general aviation airplane to sudden application of 1^0 elevator deflection – Variations of Δα and Δq over first 20 seconds.
(Adapted from Ref.1.12, chapter 6 with permission from American Institute of Aeronautics and Astronautics, Inc.)

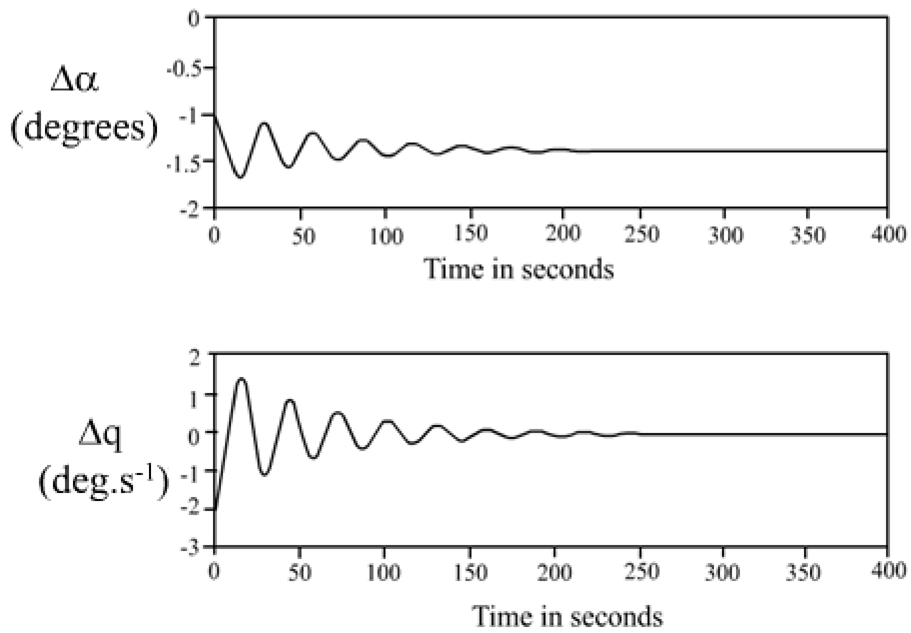

Fig.10.7b Response of the general aviation airplane to sudden application of 1^0 elevator deflection – Variations of Δα and Δq up to 400 seconds.
(Adapted from Ref.1.12, chapter 6 with permission from American Institute of Aeronautics and Astronautics, Inc.)

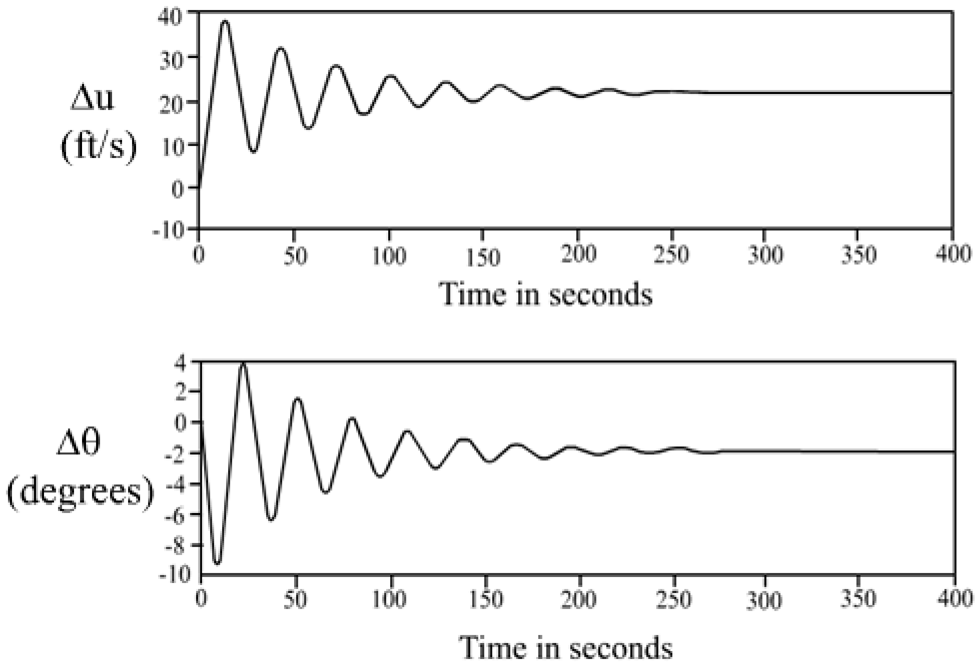

Fig.10.7c Response of general aviation airplane to sudden application of 1^0 elevator deflection – Variations of Δu and Δθ up to 400 seconds. (Adapted from Ref.1.12, chapter 6 with permission from American Institute of Aeronautics and Astronautics, Inc.)

10.5 Transfer functions for airplane motion

As noted earlier, the governing equations for the longitudinal and lateral motions are linear. Further, for the case of response to control deflection(Δδ) the values of the dependent variables (Δu, Δw etc.) would be zero at t = 0. In this case, when the Laplace transform of the governing equations are taken, a set of algebraic equation containing Laplace transforms of (a) the dependent variables [L(Δu),L(Δw) etc.] and (b) the control input [L(Δδ)] would be obtained. Solving these equations would give expressions for the ratios like [L(Δu)/L(δ)],[L(ΔW)/L(δ)] etc. These ratios are called transfer functions. A formal definition of transfer function is as follows. A transfer function is defined as the ratio of the Laplace transform of the output to the Laplace transform of the input with all initial conditions set to zero.

The transfer function concept is useful in obtaining the response of the airplane to various inputs.

To illustrate the idea of transfer function consider the longitudinal motion with the following simplifications.

(i) $\theta_0 = 0$, giving $\cos\theta_0 = 1$ and $\sin\theta_0 = 0$,

(ii) $Z_q = Z_{\dot{w}} = 0$ and (iii) $\delta_T = 0$.

The equations of motion become:

$$\left(\frac{d}{dt} - X_u\right)\Delta u - X_w\,\Delta w + g\Delta\theta = X_{\delta e}\,\Delta\delta_e \qquad (10.65)$$

$$-Z_u\,\Delta u + \left(\frac{d}{dt} - Z_w\right)\Delta w - u_0\frac{d\Delta\theta}{dt} = Z_{\delta e}\Delta\delta_e \qquad (10.66)$$

$$-M_u\,\Delta u - \left(M_{\dot{w}}\frac{d}{dt} + M_w\right)\Delta w + \frac{d}{dt}\left(\frac{d}{dt} - M_q\right)\Delta\theta = M_{\delta e}\,\Delta\delta_e \qquad (10.67)$$

Assuming that at t = 0, $\Delta u = \Delta w = \Delta\theta = 0$, taking Laplace transform and denoting $L(\Delta u) = \overline{\Delta u}$ etc. gives:

$$(s - X_u)\overline{\Delta u} - X_w\,\overline{\Delta w} + g\,\overline{\Delta\theta} = X_{\delta e}\,\overline{\Delta\delta_e} \qquad (10.68)$$

$$-Z_u\overline{\Delta u} + (s - Z_w)\overline{\Delta w} - u_0 s\,\overline{\Delta\theta} = Z_{\delta e}\,\overline{\Delta\delta_e} \qquad (10.69)$$

$$-M_u\overline{\Delta u} - (sM_{\dot{w}} + M_w)\overrightarrow{\Delta w} + s(s - M_q)\overline{\Delta\theta} = M_{\delta e}\,\overline{\Delta\delta_e} \qquad (10.70)$$

Dividing by $\overline{\Delta\delta_e}$, yields :

$$(s - X_u)\frac{\overline{\Delta u}}{\overline{\Delta\delta_e}} - X_w\frac{\overline{\Delta w}}{\overline{\Delta\delta_e}} + g\frac{\overline{\Delta\theta}}{\overline{\Delta\delta_e}} = X_{\delta e} \qquad (10.71)$$

$$-Z_u\frac{\overline{\Delta u}}{\overline{\Delta\delta_e}} + (s - Z_w)\frac{\overline{\Delta w}}{\overline{\Delta\delta_e}} - u_0 s\frac{\overline{\Delta\theta}}{\overline{\Delta\delta_e}} = Z_{\delta e} \qquad (10.72)$$

$$-M_u\frac{\overline{\Delta u}}{\overline{\Delta\delta_e}} - (sM_{\dot{w}} + M_w)\frac{\overline{\Delta w}}{\overline{\Delta\delta_e}} + s(s - M_q)\frac{\overline{\Delta\theta}}{\overline{\Delta\delta_e}} = M_{\delta e} \qquad (10.73)$$

Applying Cramer's rule to Eqs.(10.71) to (10.73) the expressions for $\frac{\overline{\Delta\theta}}{\overline{\Delta\delta_e}}, \frac{\overline{\Delta u}}{\overline{\Delta\delta_e}}, \frac{\overline{\Delta w}}{\overline{\Delta\delta_e}}$ can be written.

For example the expression for $\frac{\overline{\Delta\theta}}{\overline{\Delta\delta_e}}$ is:

$$\frac{\overline{\Delta\theta}}{\overline{\Delta\delta_e}} = \frac{\begin{vmatrix} (s - X_u) & -X_w & X_{\delta e} \\ -Z_u & (s - Z_w) & Z_{\delta e} \\ -M_u & (sM_{\dot{w}} + M_w) & M_{\delta e} \end{vmatrix}}{\begin{vmatrix} (s-X_u) & -X_w & g \\ -Z_u & (s - Z_w) & u_0 s \\ -M_u & (sM_{\dot{w}} + M_w) & s(s - M_q) \end{vmatrix}} \quad (10.74)$$

It can be written in the simplified form as:

$$\frac{\overline{\Delta\theta}}{\overline{\Delta\delta_e}} = \frac{N_\delta^\theta (s)}{\Delta_{long}} = \frac{A_\theta s^3 + B_\theta s^2 + C_\theta s + D_\theta}{\Delta_{long}} \quad (10.75)$$

Similarly

$$\frac{\overline{\Delta u}}{\overline{\Delta\delta_e}} = \frac{N_\delta^u (s)}{\Delta_{long}} = \frac{A_u s^3 + B_u s^2 + C_u s + D_u}{\Delta_{long}} \quad (10.76)$$

$$\frac{\overline{\Delta w}}{\overline{\Delta\delta_e}} = \frac{N_\delta^w (s)}{\Delta_{long}} = \frac{A_w s^3 + B_w s^2 + C_w s + D_w}{\Delta_{long}} \quad (10.77)$$

$$\Delta_{long} = As^4 + Bs^3 + Cs^2 + Ds + E \quad (10.78)$$

Where A, B, C, D and E in Eq.(10.78) are given by Eq.(8.10). The expressions for A_θ, B_θ, C_θ and D_θ can be worked out from Eq.(10.74) . See also Ref.1.1, chapter 8.

Similarly, taking Laplace transform of the governing equations for the lateral motion, yields:

$$(s - Y_v)\frac{\overline{\Delta v}}{\overline{\Delta\delta}} - (u_0 - Y_r)\frac{\overline{\Delta r}}{\overline{\Delta\delta}} - g\frac{\overline{\Delta\varphi}}{\overline{\Delta\delta}} = Y_\delta \quad (10.79)$$

$$-L'_v\frac{\overline{\Delta v}}{\overline{\Delta\delta}} + (s^2 - L'_p s)\frac{\overline{\Delta\varphi}}{\overline{\Delta\delta}} - [\frac{I_{xz}}{I_{xx}}s^2 + L'_r s]\frac{\overline{\Delta\psi}}{\overline{\Delta\delta}} = L'_\delta \quad (10.80)$$

$$-N_v\frac{\overline{\Delta v}}{\overline{\Delta\delta}} - (\frac{I_{xz}}{I_{zz}}s^2 + N_p s)\frac{\overline{\Delta\varphi}}{\overline{\Delta\delta}} - [s^2 - N_r s]\frac{\overline{\Delta\psi}}{\overline{\Delta\delta}} = N_\delta \quad (10.81)$$

where, $r = \dot{\psi}$, $p = \dot{\varphi}$ and $\overline{\Delta\delta}$ could be either $\overline{\Delta\delta_a}$ or $\overline{\Delta\delta_r}$ (10.82)

Again applying Cramer's rule, the set of Eqs.(10.79) to (10.81)can be written as:

$$\frac{\overline{\Delta v}}{\overline{\Delta\delta}} = \frac{A_v s^4 + B_v s^3 + C_v s^2 + D_v s}{As^5 + Bs^4 + Cs^3 + Ds^2 + Es} \quad (10.83)$$

Or $$\frac{\overline{\Delta v}}{\overline{\Delta \delta}} = \frac{A_v s^3 + B_v s^2 + C_v s + D_v}{As^4 + Bs^3 + Cs^2 + Ds + E} \tag{10.84}$$

For further details see Ref.1.1, chapter 8.

10.6 Concluding remarks

In this course on flight Dynamics II, chapter 1 dealt with the concepts of stability and control and the division of the course content into various subtopics.

Chapters 2, 3 and 4 dealt with the static longitudinal stability in steady flight with controls fixed, control free and during manoeuvres.

Chapters 5 and 6 dealt with the static directional and lateral stability and control.

The chapters 7, 8 and 9 were devoted to longitudinal and lateral dynamic stability.

The tenth chapter touched upon stability after stall, automatic control, response of two degrees of freedom systems and transfer functions.

The aim of the course is to give enough basic knowledge on airplane stability and control, so that the reader is able to pursue further studies on his own. The reader is advised to study Ref.1.1,1.12,1.13, 7.2 and 10.3 for further details on topics like (a) airplane response to control and atmospheric inputs (b) automatic control theory and design of autopilot (c) theory and design of linear systems (d) stability and control at high angles of attack and (e) inertia coupling and spin.