Chapter 9

Dynamic stability analysis – III – Lateral motion

(Lectures 33 and 34)

Keywords : Lateral dynamic stability - state variable form of equations, characteristic equation and its roots ; motions indicated by roots - role subsidence, spiral mode, Dutch roll ; stability diagram.

Topics

Chapter 9

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Lecture 33

Topics

9.1 Introduction

In chapter 7 the small perturbation equations for the longitudinal and lateral motions with controls fixed were derived. Expressions for the stability derivatives were also obtained. The analysis of the longitudinal dynamic stability was dealt with in Chapter 8. With this background, the lateral dynamic stability is briefly discussed in this chapter. The distinguishing features of lateral dynamic stability are highlighted.

9.2 State variable form of lateral dynamic stability equation

The equations of motion for the lateral case (Eqs.7.88, 7.89, 7.90) are reproduced here for ready reference.

$$\left(\frac{d}{dt}-Y_v\right)\Delta v-(u_0-Y_r)\Delta r-g\cos\theta_0\,\Delta\varphi=Y_{\delta r}\,\Delta\delta_r \tag{7.88}$$

$$-L'_v\,\Delta v+\left(\frac{d}{dt}-L'_p\right)\Delta p-\left[\frac{I_{xz}}{I_{xx}}\frac{d}{dt}+L'_r\right]\Delta r$$

$$=L'_{\delta a}\,\Delta\delta_a+L'_{\delta r}\,\Delta\delta_r \tag{7.89}$$

$$- N_v \Delta v - (\frac{I_{xz}}{I_{zz}}\frac{d}{dt} + N_p)\Delta p - [\frac{d}{dt} - N_r]\Delta r$$

$$= N_{\delta a}\ \Delta\delta_a + N_{\delta r}\ \Delta\delta_r \qquad (7.90)$$

These equations can be put in the state variable form using the following steps:

$$\Delta\dot{v} = Y_V\ \Delta v + Y_p \Delta p - (u_0 - Y_r)\Delta r + g\cos\theta_0 \Delta\varphi + Y_{\delta r}\ \Delta\delta_r \qquad (9.1)$$

$$\Delta\dot{p} = L'_V\ \Delta v + L'_p\ \Delta p + \frac{I_{XZ}}{I_{XX}}\frac{d}{dt}\ \Delta r + L'_r\ \Delta r + L'_{\delta a}\Delta\delta_a + L'_{\delta r}\ \Delta\delta_r \qquad (9.2)$$

$$\Delta\dot{r} = N_V\,\Delta v + N_p \Delta p + \frac{I_{XZ}}{I_{ZZ}}\frac{d}{dt}\Delta p + N_r\,\Delta r + N_{\delta a}\Delta\delta_a + N_{\delta r}\,\Delta\delta_r \qquad (9.3)$$

$$\Delta\dot{\varphi} = \Delta p \qquad (9.4)$$

The term [d(Δr) / dt] or $\Delta\dot{r}$ occurs on the right hand side of Eq.(9.2). Its expression, given by Eq.(9.3), is used to replace it. The resulting equation would have the following term on the right hand side:

$$\frac{I_{xz}^2}{I_{xx}I_{zz}}\frac{d}{dt}\Delta p$$

Taking it to the left hand side yields:

$$(1 - \frac{I_{xz}^2}{I_{xx}\,I_{zz}})\frac{d}{dt}\Delta p = (L'_v + \frac{I_{xz}}{I_{xx}}N_v)\Delta v + (L'_p + \frac{I_{xz}}{I_{xx}}N_p)\Delta p$$

$$+ (L'_r + \frac{I_{xz}}{I_{xx}}N_r)\Delta r + (L'_{\delta a} + \frac{I_{xz}}{I_{xx}}N_{\delta a})\Delta\delta_a + (L'_{\delta r} + \frac{I_{xz}}{I_{xx}}N_{\delta r})\Delta\delta_r \qquad (9.5)$$

The following notations are introduced here (Ref.1.1, chapter 4) :

$$L_v^* = \frac{L'_v}{(1 - \frac{I_{xz}^2}{I_{xx}I_{zz}})},\ N_v^* = \frac{N_v}{(1 - \frac{I_{xz}^2}{I_{xx}I_{zz}})},\ \text{etc.} \qquad (9.6)$$

Use of the notations given in Eq.(9.6), renders Eq.(9.5) as:

$$\Delta\dot{p} = (L_v^* + \frac{I_{xz}}{I_{xx}}N_v^*)\Delta v + (L_p^* + \frac{I_{xz}}{I_{xx}}N_p^*)\ \Delta p$$

$$+ (L_r^* + \frac{I_{xz}}{I_{xx}}N_r^*)\Delta r + (L_{\delta a}^* + \frac{I_{xz}}{I_{xx}}N_{\delta a}^*)\ \Delta\delta_a + (L_{\delta r}^* + \frac{I_{xz}}{I_{xx}}N_{\delta r}^*)\Delta\delta_r \qquad (9.7)$$

Similarly, eliminating $\Delta\dot{p}$ from r.h.s of Eq. (9.3) and simplifying gives the following expression for $\Delta\dot{r}$:

$$\Delta\dot{r} = (N_v^* + \frac{I_{xz}}{I_{zz}} L_v^*)\Delta v + (N_p^* + \frac{I_{xz}}{I_{zz}} L_p^*)\Delta p$$

$$+ (N_r^* + \frac{I_{xz}}{I_{zz}} L_r^*)\Delta r + (N_{\delta a}^* + \frac{I_{xz}}{I_{zz}} L_{\delta a}^*)\Delta\delta_a + (N_{\delta r}^* + \frac{I_{xz}}{I_{zz}} L_{\delta r}^*)\Delta\delta_r \qquad (9.8)$$

The final form of the equations for lateral motion in the state variable form is:

$$\begin{bmatrix}\Delta\dot{v}\\ \Delta\dot{p}\\ \Delta\dot{r}\\ \Delta\dot{\varphi}\end{bmatrix} = \begin{bmatrix} Y_v & Y_p & -u_0+Y_r & g\cos\theta_0 \\ L_v^* + \frac{I_{xz}}{I_{xx}}N_v^* & L_p^* + \frac{I_{xz}}{I_{xx}}N_p^* & L_r^* + \frac{I_{xz}}{I_{xx}}N_r^* & 0 \\ N_v^* + \frac{I_{xz}}{I_{zz}}L_v^* & N_p^* + \frac{I_{xz}}{I_{zz}}L_p^* & N_r^* + \frac{I_{xz}}{I_{zz}}L_r^* & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix}\Delta v\\ \Delta p\\ \Delta r\\ \Delta\varphi\end{bmatrix}$$

$$+ \begin{bmatrix} 0 & Y_{\delta_r} \\ L_{\delta a}^* + \frac{I_{xz}}{I_{xx}}N_{\delta a}^* & L_{\delta r}^* + \frac{I_{xz}}{I_{xx}}N_{\delta_r}^* \\ N_{\delta a}^* + \frac{I_{xz}}{I_{zz}}L_{\delta a}^* & N_{\delta r}^* + \frac{I_{xz}}{I_{zz}}L_{\delta r}^* \end{bmatrix} \begin{bmatrix}\Delta\delta_a\\ \Delta\delta_r\end{bmatrix} \qquad (9.9)$$

This can be put in the form:

$$\dot{\mathbf{X}} = \mathbf{A.X} + \mathbf{B.\eta} \qquad (9.10)$$

When the control deflections (δ_a and δ_r in the present case) are fixed, $\boldsymbol{\eta} = 0$ and Eq.(9.10) reduces to:

$$\dot{\mathbf{X}} = \mathbf{A.X} \qquad (9.11)$$

As pointed out in section 8.10, Eq(9.11) has the following solution.

$$|\lambda_r \mathbf{I} - \mathbf{A}| = 0 \qquad (9.12)$$

Where λ_r' s are the eigen values of matrix **A**.

Expanding Eq.(9.12), the following characteristic equation for the lateral motion is obtained.

$$A_1 \lambda^4 + B_1 \lambda^3 + C_1 \lambda^2 + D_1 \lambda + E_1 = 0 \quad (9.13)$$

9.3 Roots of stability quartic for general aviation airplane (Navion)

This section illustrates the process of setting up matrix **A** and then obtaining the eigen values.

Example 9.1

The general aviation airplane (Navion) discussed in example 8.1 has the following lateral stability derivatives (Ref.2.4, section 'X').

Y_v = - 0.2543 s^{-1},Y_β = -13.64 m s^{-2}, Y_p = 0, Y_r = 0, L'_v = -0.298 m^{-1} s^{-1},

L'_β = -15.962 s^{-2}, L'_p = - 8.402 s^{-1}, L'_r = 2.193 s^{-1}, N_v = 0.0838 $m^{-1}s^{-1}$,

N_β = 4.495 s^{-2} , N_p = - 0.3498 s^{-1}, N_r = - 0.7605 s^{-1}, I_{xx} = 1420.9 kg m^2 ,

I_{zz} = 4786. 0 kg m^2, I_{xz} = 0.

The matrix **A** in Eqs.(9.9) and (9.10) in this case is

$$\mathbf{A} = \begin{bmatrix} -0.2543 & 0 & -53.64 & -9.80665 & 0 \\ -0.298 & -8.402 & 2.193 & 0 & 0 \\ 0.0838 & -0.3498 & -0.7608 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad (9.14)$$

The eigen values of this matrix are the roots of the lateral stability quatric. Using MATLAB software the roots are:

- 0.0089, - 8.4345, - 0.4867 ± i 2.3314

9.3.1 Iterative solution of lateral stability quartic

The characteristic equation (Eq. 9.13) is called lateral stability quartic. Reference 1.5, part III chapter 2, contains an iterative method to solve it. The case of Navion is considered to illustrate the procedure. Using the stability derivatives in example 9.1, the following characteristic equation is obtained.

$$\lambda^4 + 9.4168\lambda^3 + 13.9662\lambda^2 + 47.9666\lambda + 0.4258 = 0 \quad (9.15)$$

As mentioned in section 8.5, the iterative technique, for obtaining roots of a quartic, depends on the relative magnitudes of the coefficients of the terms in the polynomial. For lateral stability quartic a technique different from that used in section 8.5 is employed.

It is seen that the coefficient E_1 is much smaller than D_1. Hence, a small root is expected. When the root is small, the terms with powers of λ would be very small in Eq.(9.15) and the first approximation $\lambda_1^{(1)}$ can be written as:

$$\lambda_1^{(1)} = - E_1 / D_1 = 0.4258 / 47.9666 = 0.00888$$

To get the second approximation to λ_1, Eq.(9.15) is written as:

$$(\lambda+0.00888)\,(\lambda^3+9.4079\lambda^2+13.8827\lambda+47.8433)+0.003651=0$$

If $\lambda_1^{(1)}$ were the exact root, then the remainder should be zero. In this case it is 0.003651.

The second approximation is obtained as:

$$\lambda_1^{(2)} = - 0.4258 / 47.8433 = - 0.0089$$

Substituting in Eq.(9.15), gives:

$$(\lambda+0.0089)(\lambda^3+9.4079\,\lambda^2+13.8825\,\lambda+47.8430)+2.7\times 10^{-6} \qquad (9.16)$$

Since, the remainder is very small λ_1 = - 0.0089 is taken as the first root.

To get the other three roots, the cubic part of Eq.(9.16) is now considered i.e.

$$\lambda^3+9.4079\lambda^2+13.8825\lambda+47.843 = 0 \qquad (9.17)$$

To solve the Eq.(9.17), it is assumed that the second root (λ_2) is large. Then, for the first approximation ($\lambda_2^{(1)}$), the second and third terms in Eq.(9.17) can be ignored i.e.

$$\lambda^3+9.4079\lambda^2 = 0 \text{ or } \lambda_2^{(1)} = - 9.4079$$

To get the second approximation to λ_2, Eq.(9.17) is divided by λ^2 and $\lambda_2^{(1)}$ is substituted in the neglected terms. This gives :

$$\lambda_2^{(2)} = - 9.4079 - \frac{13.8825}{(-9.4079)} - \frac{47.893}{(-9.4079)^2} = - 9.4728$$

In a similar fashion, $\lambda_2^{(3)} = - 8.4358$ and $\lambda_2^{(4)} = - 8.4345$ are obtained. The last two values are fairly close to each other and $\lambda_2 = - 8.4345$ is taken as the second root.

Substituting for λ_1 and λ_2 in Eq.(9.15) leads to:

$(\lambda+0.0089)(\lambda+8.4345)(\lambda^2+0.9734\,\lambda+5.6726)$.

Hence, the four roots are: - 0.0089, - 8.4345, -0.4867 $\pm$ i 2.3315

which are almost the same as those obtained by using matrix '**A**' and the MATLAB Software.

9.4 Lateral dynamic stability analysis which includes angular displacement in yaw (ΔΨ) as a state variable

In the analysis presented in section 9.2, the quantity Δφ is included as a variable since, it is directly involved in Eq.(7.88). The angular displacement in yaw ($\Delta\psi$) can also be included. In this case the following additional governing equation is obtained.

$$\Delta\dot{\psi} = r \tag{9.18}$$

Reference 1.12 (chapter 6) and Ref.1.5 (chapter 7 part II) follow this approach. In this case Eq.(9.9) would get modified as:

$$\begin{bmatrix} \Delta\dot{v} \\ \Delta\dot{p} \\ \Delta\dot{r} \\ \Delta\dot{\varphi} \\ \Delta\dot{\psi} \end{bmatrix} = \begin{bmatrix} Y_v & Y_p & -u_0+Y_r & g\cos\theta_0 & 0 \\ L_v^* + \frac{I_{xz}}{I_{xx}} N_v^* & L_p^* + \frac{I_{xz}}{I_{xx}} N_p^* & L_r^* + \frac{I_{xz}}{I_{xx}} N_r^* & 0 & 0 \\ N_v^* + \frac{I_{xz}}{I_{zz}} L_v^* & N_p^* + \frac{I_{xz}}{I_{zz}} L_p^* & N_r^* + \frac{I_{xz}}{I_{zz}} L_r^* & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta\varphi \\ \Delta\psi \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & Y_{\delta r} \\ L_{\delta a}^* + \frac{I_{xz}}{I_{xx}} N_{\delta a}^* & L_{\delta_r}^* + \frac{I_{xz}}{I_{xx}} N_{\delta r}^* \\ N_{\delta a}^* + \frac{I_{xz}}{I_{zz}} L_{\delta a}^* & N_{\delta r}^* + \frac{I_{xz}}{I_{zz}} L_{\delta r}^* \end{bmatrix} \begin{bmatrix} \Delta\delta_a \\ \Delta\delta_r \end{bmatrix} \tag{9.19}$$

The characteristic equation corresponding to this set of equations would be :

$$A_1\lambda^5 + B_1\lambda^4 + C_1\lambda^3 + D_1\lambda^2 + E_1\lambda = 0 \tag{9.20}$$

This equation has λ = 0 as the fifth root in addition to the four roots discussed earlier.

Note:

Similar result (i.e. a fifth degree polynomial as the characteristic equation), would have been obtained, if Eqs.(7.88),(7.89) and (7.90) were solved assuming :

$$\Delta v = \rho_1 e^{\lambda t},\ \Delta\varphi = \rho_2 e^{\lambda t},\ \Delta\psi = \rho_3 e^{\lambda t} \qquad (9.21)$$

9.5. Response indicated by roots of characteristic equation of lateral motion – change of flight direction, roll subsidence, spiral mode and Dutch roll

The characteristic equation for the lateral motion (Eq.9.20) has five roots. The motions indicated by these can be briefly described as follows.

I) λ equal to zero root:

From Eq.(9.21) it is observed that when, λ is zero the solution would be:

$$\Delta v = \rho_1,\ \Delta\varphi = \rho_2,\ \Delta\psi = \rho_3. \qquad (9.22)$$

Substituting these in Eqs.(7.88),(7.89) and (7.90) shows that Δv equals 0 but Δφ and ΔΨ need not be zero. Thus the zero root, which indicates neutral stability, could lead to the airplane attaining a constant angle of bank (Δφ) and / or yaw (ΔΨ). Figure 9.1 shows a change in flight direction as a result of the zero root. This feature is discussed again, in section 9.6 , when the response of the general aviation airplane to a lateral disturbance is considered . It may be pointed out that appropriate control action is needed to correct the constant values of Δφ and Δψ .

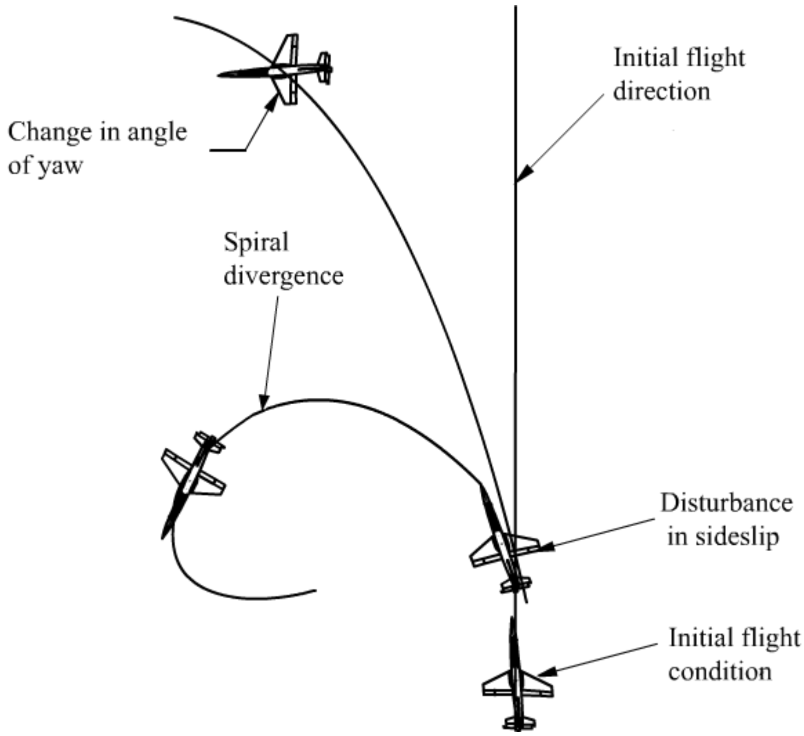

Fig.9.1 Directional divergence and spiral divergence

II) Large negative root:

It is called roll subsidence. It is heavily damped and does not pose any problem.

III) Small negative root:

The lightly damped real root is called spiral mode. When the root is positive i.e. unstable, the airplane loses altitude, gains speed, banks more and more with increasing turn rate. The flight path is a slowly tightening spiral motion (Fig.9.1). The difference between spiral motion and spin is that in the former the angle of attack is below stall and the control surfaces are effective. Section 10.2 briefly deals with spin.

IV) The complex root indicating oscillatory motion:

The motion corresponding to this root is called Dutch roll. It can be briefly described as follows.

a) Let the disturbance cause roll to left.
b) Due to adverse yaw, this roll results in yaw to right, i.e. sideslip to the right.

c) Because of the dihedral effect, the sideslip causes a restoring moment causing roll to right.
d) Subsequently airplane yaws to left and rolls to left.
e) This sequence of events continues and when the real part of the root is negative the amplitude of the oscillatory motion decreases with time.
f) During this oscillatory motion the airplane is all along moving forward (Fig.9.2). The motion is similar to the weaving motion of an ice skater. This mode is called Dutch roll.

   According to Wikipedia, skating repetatively to the right and to the left is referred to as Dutch roll. J.Hunsaker appears to be the first to refer the aforesaid oscillatory motion of the airplane as Dutch roll.

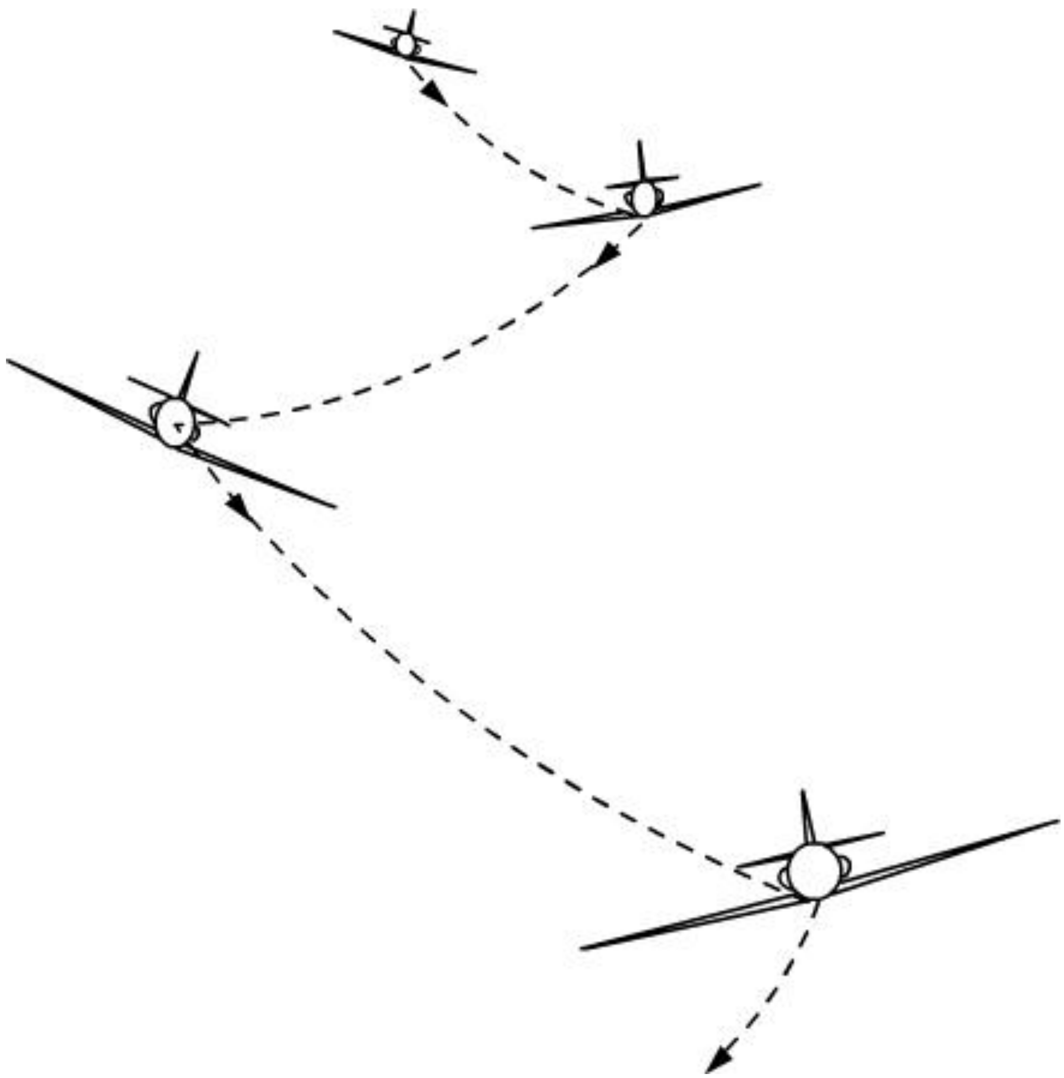

Fig. 9.2 Dutch roll

Remark:

Dutch roll is an undesirable mode as it causes discomfort to the passengers in civil airplanes and may result in missing the enemy targets in military airplanes. It should have adequate damping.