Chapter 8

Dynamic stability analysis – II – Longitudinal motion - 5

Lecture 32

Topics

8.14 Eigen values and Eigen vectors

As mentioned in section 8.10, a linear set of equations can be expressed as:

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{B\eta} \quad (8.34)$$

Where, **X** is the state vector and **η** is the control vector. Further, in stability analysis, matrices **A** and **B** contain the stability derivatives. When **η** is zero, the set reduces to:

$$\dot{\mathbf{X}} = \mathbf{AX} \quad (8.43)$$

This set of equations has the solution.

$$\mathbf{X} = \mathbf{X}_r e^{\lambda_r t} \quad (8.44)$$

Where, λ_r's are the eigen values of the matrix **A**.

8.14.1. Eigen vector

If λ_j is an Eigen values of a square matrix **A**, then a non-zero vector **X** which satisfies the following equation is called eigen vector corresponding to the eigen value λ_j.

$$\mathbf{AX} = \lambda_j \mathbf{X} \quad (8.66)$$

Remarks:

i) Each eigen value has an eigen vector associated with it.

ii) Eigen values decide the nature of the motion following the disturbance and eigen vectors indicate the amplitude of the response. The approach discussed in

Ref.1.12, chapter 5, is used here to explain these aspects with the help of an example.

Consider a two degree of freedom system governed by the following set of equations *.

$$3\dot{x}_1+2x_1+\dot{x}_2 = 0 \quad (8.67)$$

$$\dot{x}_1+4\dot{x}_2+3x_2 = 0 \quad (8.68)$$

By simple manipulation, this system, can be expressed in state variable form as:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\frac{1}{11}\begin{bmatrix}-8 & 3\\ 2 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \quad (8.69)$$

The eigen values of the square matrix in Eq.(8.69) are given by:

$|\boldsymbol{\lambda}\mathbf{I}-\mathbf{A}|=0$, where $\mathbf{I}$ is the identity matrix.

Or $$\begin{vmatrix}\lambda+\frac{8}{11} & -\frac{3}{11}\\ -\frac{2}{11} & \lambda+\frac{9}{11}\end{vmatrix} = 0$$

Expanding and simplifying gives:

$$11\lambda^2 + 17\lambda + 6 = 0$$

which has the roots, $\lambda = -1, -6/11$

The eigen vectors corresponding to these two eigen values are obtained as follows.

For $\lambda = -1$, Eq.(8.66) gives:

$$\frac{1}{11}\begin{bmatrix}-8 & 3\\ 2 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=-1\begin{bmatrix}x_1\\ x_2\end{bmatrix} \quad (8.70)$$

Or $-8x_1 + 3x_2 = -11x_1$ (8.71)

and $2x_1 - 9x_2 = -11x_2$ (8.72)

Simplifying Eqs.(8.71) and (8.72) gives:

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*Taken from section 7,chapter 6 of Ref 8.3, with permission from McGraw-Hill Book Co.

$$3x_1 + 3x_2 = 0$$

and $$2x_1 + 2x_2 = 0$$

Both of these are the same. This means that the eigen vector is not unique and depends on the choice of one of the two variables. Choosing $x_2 = 1$, gives $x_1 = -1$. Thus, the eigen vector corresponds to $\lambda = -1$ is

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ or } [\,-1 \quad 1\,]^T \qquad (8.73)$$

Simplifying yields respectively:

$$-2x_1 + 3x_2 = 0$$

$$2x_1 - 3x_2 = 0$$

Both of these are the same. Choosing $x_2 = 1$ gives $x_1 = 3/2$. Hence, the eigen vector corresponding to $\lambda = -6/11$ is $[\,3/2 \ 1\,]^T$. (8.74)

To clarify the physical significance of the Eigen values and vectors, consider the solution as (recall section 8.2):

$$x_1(t) = \rho_1 e^{\lambda t} \text{ and } x_2(t) = \rho_2 e^{\lambda t} \qquad (8.75)$$

Substituting in the governing equations (i.e. Eq.8.67 and 8.68) yields :

$$3\lambda \rho_1 + 2\rho_1 + \lambda \rho_2 = 0$$

$$\lambda \rho_1 + 4\lambda \rho_2 + 3\rho_2 = 0$$

Which can be expressed as:

$$\lambda \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -8 & 3 \\ 2 & -9 \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \qquad (8.76)$$

Comparing Eqs.(8.69) and (8.76) it is noted that the square matrix is the same in both equations as it should be. Hence, the eigen values of matrix in Eq.(8.76) are -1 and - 6/11 and the corresponding eigen vectors are $[-1 \ \ 1]^T$ and $[3/2 \ \ 1]^T$ respectively. This indicates that for $\lambda = -1$, $\rho_1 = -1$ and $\rho_2 = 1$. Then, the first mode given by the governing equations is :

$$x_1 = -e^{-t} \text{ and } x_2 = e^{-t}$$

With $\lambda = -6/11$, $\rho_1 = 3/2$ and $\rho_2 = 1$. Then, the second mode given by the governing equations is:

$$x_1 = \frac{3}{2} e^{-\frac{6}{11}t} \qquad (8.77)$$

and $$x_2 = e^{-\frac{6}{11}t} \qquad (8.78)$$

From Eqs.(8.77 and 8.78) it is evident that the eigen values indicate the nature of the motion following the disturbance and eigen vectors indicate the amplitude of the response.

Note: Actual responses of this system to a chosen disturbance and control input are given in sections 10.4.2 to 10.4.4.

8.14.2 Eigen vectors for Navion

Examples 8.1 and 8.2 present the stability derivatives and roots of the characteristic equation for the general aviation airplane (Navion) flying at sea level at a flight velocity of 53.64 m/s. The roots are:

Short period oscillation (SPO):- 2.508 ± i 2.577;

Phugoid or long period oscillation (LPO): - 0.01715 ± i 0.2135.

To obtain the Eigen vectors for these roots the steps given in Ref.1.1, chapter 4 are followed.

I) In the state space variable form the state variables in this case are Δu, Δw, Δq and $\Delta\theta$. The governing equation in matrix form are (Eq.8.40):

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{w} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} X_u & X_w & 0 & -g \\ Z_u & Z_w & u_0 & 0 \\ M_u + M_{\dot{w}} Z_u & M_w + M_{\dot{w}} Z_w & M_q + M_{\dot{w}} u_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix} \qquad (8.40)$$

II) Following Ref.1.1, chapter 4, elements of matrix **A** are denoted by A_{11}, A_{12}…,A_{43}, A_{44} , instead of X_u,X_w... 1, 0. Further, let a root of the characteristic equation be denoted by λ_j . Let, the eigen vector corresponding to this root be denoted by $[\,\Delta u_j \;\; \Delta w_j \;\; \Delta q_j \;\; \Delta\theta_j\,]^T$. Then, applying Eq.(8.66) to this case yields :

$$\begin{aligned}
&(\lambda_j - A_{11})\Delta u_j - A_{12}\,\Delta w_j - A_{13}\,\Delta q_j - A_{14}\,\Delta\theta_j = 0\\
&-A_{21}\Delta u_j + (\lambda_j - A_{22})\Delta w_j - A_{23} q_j - A_{14}\,\Delta\theta_j = 0\\
&-A_{31}\Delta u_j - A_{32}\Delta w_j + (\lambda_j - A_{33})\Delta q_j - A_{34}\,\Delta\theta_j = 0\\
&-A_{41}\Delta u_j - A_{42}\Delta w_j - A_{43}\Delta q_j + (\lambda_j - A_{44})\Delta\theta_j = 0
\end{aligned} \tag{8.79}$$

In the set of equations represented by Eq.(8.79) the stability derivatives A_{11},A_{12}..A_{44} and the eigen value λ_j are known quantities. The solution of this set gives the eigen vector namely Δu_j, Δw_j, Δq_j and $\Delta\theta_j$. As noted in the previous section, the eigen vector is not unique and out of its four elements,in the present case, three of them should be expressed in terms of the fourth one. Let, Δu_j, Δw_j and Δq_j be expressed in terms of $\Delta\theta_j$. Dividing the first three equations of Eq.(8.79) by $\Delta\theta_j$ and dropping the fourth equation yields:

$$\begin{aligned}
&(\lambda_j - A_{11})\left(\frac{\Delta u}{\Delta\theta}\right)_j - A_{12}\left(\frac{\Delta w}{\Delta\theta}\right)_j - A_{13}\left(\frac{\Delta q}{\Delta\theta}\right)_j = A_{14}\\
&-A_{21}\left(\frac{\Delta u}{\Delta\theta}\right)_j + (\lambda_j - A_{22})\left(\frac{\Delta w}{\Delta\theta}\right)_j - A_{23}\left(\frac{\Delta q}{\Delta\theta}\right)_j = A_{24}\\
&-A_{31}\left(\frac{\Delta u}{\Delta\theta}\right)_j - A_{32}\left(\frac{\Delta w}{\Delta\theta}\right)_j + (\lambda_j - A_{33})\left(\frac{\Delta q}{\Delta\theta}\right)_j = A_{34}
\end{aligned} \tag{8.80}$$

The set of Eq.(8.80) when solved by standard techniques yields the eigen vector $[\frac{\Delta u}{\Delta\theta}\ \ \frac{\Delta w}{\Delta\theta}\ \ \frac{\Delta q}{\Delta\theta}\ \ 1]^T$.

The results are presented in Table 8.2. The following may be noted.

(a) In table 8.2 the Eigen vectors are in non-dimensional form i.e.

$$\frac{\Delta u / u_0}{\Delta\theta},\ \frac{\Delta w / u_0}{\Delta\theta},\ \frac{\Delta[q\bar{c}/(2u_0)]}{\Delta\theta}$$

Eigen vector element	Long period oscillation	Short period oscillation
	$\lambda = -0.01709 \pm i\,0.2124$	$\lambda = -2.5085 \pm i\,2.5931$
$\frac{\Delta u/u_0}{\Delta\theta}$	$-0.1194 \pm i\,0.8437$	$0.0328 \pm i\,0.0235$
$\frac{\Delta w/u_0}{\Delta\theta}$	$0.008136 \pm i\,0.05027$	$1.139 \pm i\,0.7574$
$\frac{\Delta[q\bar{c}/(2u_0)]}{\Delta\theta}$	$-0.0002767 \pm i\,0.00344$	$-0.0406 \pm i\,0.04198$

Table 8.2 Eigen vector for general aviation airplane - longitudinal motion

(b) The elements of eigen vector are complex as the roots are complex. The elements of the eigen vectors give idea about the relative magnitudes of the motion variables in the corresponding mode. When these ratios are complex numbers, as happens in the present case, they can be plotted in a vector or Argand diagram. Figure 8.9a and 8.9b present the information for LPO and SPO. From Fig.8.9a and Table 8.2 it is observed that the real parts of

$\frac{\Delta w/\Delta u_0}{\Delta\theta}$ and $\frac{\Delta(q\bar{c}/2u_0)}{\Delta\theta}$ are very small and are not seen in the diagram.

The magnitude of $\frac{\Delta u/u_0}{\Delta\theta}$ is $\sqrt{0.1194^2 + 0.8437^2}$ or 0.8521.

The phase of $\frac{\Delta u/u_0}{\Delta\theta}$ is given by $\tan^{-1}(-0.8437/0.1194) = 98.05^0$

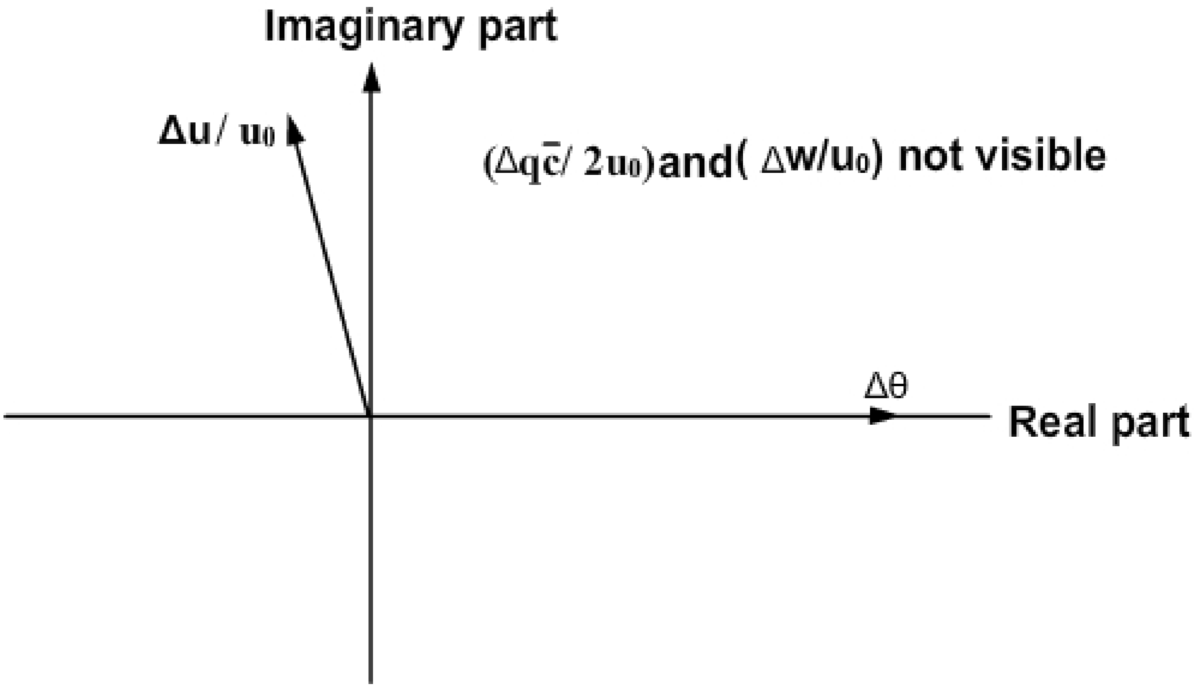

Table 8.9a Argand diagram for Eigen vectors of LPO

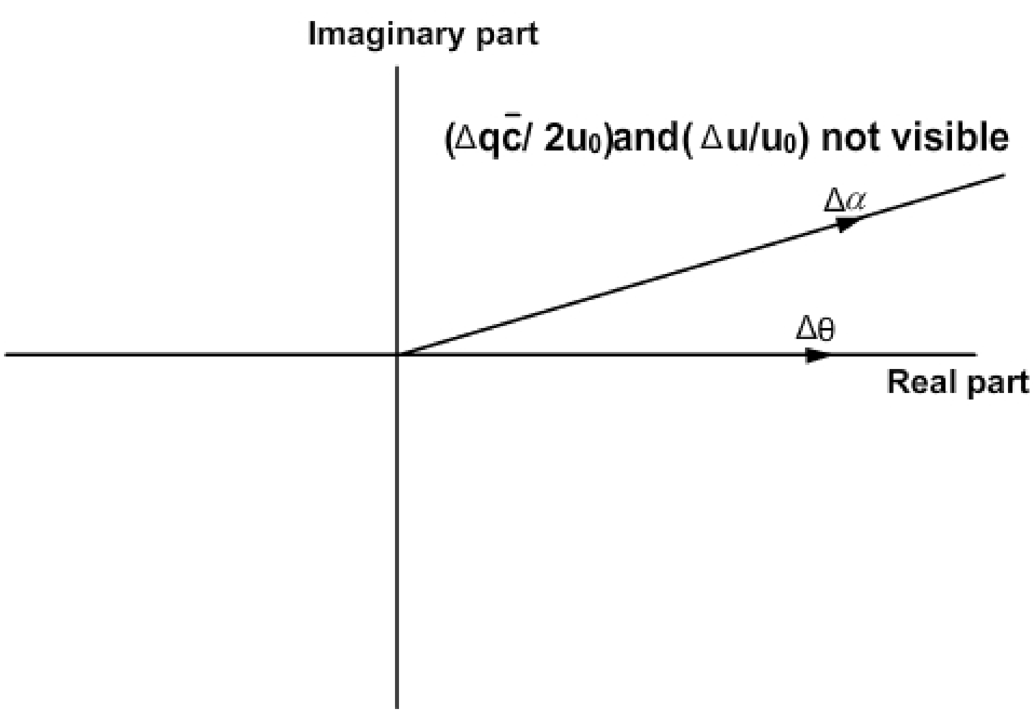

Table 8.9b Argand diagram for Eigen vectors of SPO

From Fig.8.9b and table 8.2, it is seen that for SPO $(\frac{\Delta u/u_0}{\Delta\theta})$ and $(\frac{\Delta qc/2u_0}{\Delta\theta})$ are very small. But, $(\frac{\Delta w/u_0}{\Delta\theta})$ has magnitude of $\sqrt{1.139^2+0.7574^2}=1.3678$ and phase of $\tan^{-1}\frac{0.7574}{1.139}=33.62^0$.

From the above discussion the following conclusions can be drawn. They were also drawn earlier, while simplifying the equations for LPO and SPO in section 8.11.

(i) The SPO is characterized by negligible changes in flight speed. The angle of attack oscillates with amplitude and phase not significantly different from Δθ.

(ii) As regards LPO, the changes in Δq and Δα are very small. Δu has significant magnitude and leads Δθ by about 90^0.

8.15 Longitudinal stick-free dynamic stability

While defining the degrees of freedom it was mentioned that they are the number of coordinates needed to prescribe the position of any point on the system. A rigid airplane with controls fixed has six degrees of freedom viz. the three coordinates of the c.g. with respect to a ground fixed axis system and three Eulerian angles. When the elevator is free to rotate about its hinge, an additional degree of freedom is introduced i.e. to describe the position of a point on the elevator, the elevator deflection (δ_e) needs to be prescribed; note that the elevator is still assumed to be rigid.

The rotation of the elevator causes changes in the aerodynamic forces and the moments about the c.g.. Noting that $(M_{cg})_{\delta e} = M_{\delta e}\Delta\delta_e + M_{\dot{\delta} e}\ \Delta\dot{\delta}_e$. The third of the governing equations (Eq.7.87) becomes:

$$- M_u\Delta u - (M_{\dot{w}}\frac{d}{dt}+M_w)\Delta w + (\frac{d^2}{dt^2} - M_q\frac{d}{dt})\Delta\theta - M_{\delta e}\Delta\delta_e + M_{\dot{\delta} e}\Delta\dot{\delta}_e \qquad (8.81)$$

The new equation for the motion of elevator about it's hinge would look like (See Ref.1.7, chapter 10).

$$\frac{\partial H}{\partial u}\Delta u + \frac{\partial H}{\partial \alpha}\Delta\alpha + \frac{\partial H}{\partial \delta_e}\Delta\delta_e + \frac{\partial H}{\partial \dot{\theta}}\Delta\dot{\theta} + \frac{\partial H}{\partial \dot{\alpha}}\Delta\dot{\alpha} + \frac{\partial H}{\partial \dot{\delta}_e}\dot{\delta}_e$$

$$= I_e(\ddot{\theta} + \ddot{\delta}_e) + m_e x_e(u_0\dot{\alpha} - u_0\dot{\theta} + l_t\ddot{\theta}) \qquad (8.82)$$

Where, H = hinge moment, m_e = mass of the elevator, I_e = moment of inertia of the elevator and x_e = distance of elevator c.g. behind the hinge line.

Thus, the system of equation with elevator free will consists of four governing equations namely Eqs.(7.85), (7.86), (8.81) and (8.82).

The characteristic equation for this set of equations would be of the form:

$$A\lambda^6 + B\lambda^5 + C\lambda^4 + D\lambda^3 + E\lambda^2 + F\lambda + G = 0. \qquad (8.83)$$

where, G depends on $(dC_m/dC_L)_{\text{stick-free}}$.

It is found that the six roots of Eq.(8.83) form three complex pairs. Out of these three pairs, the first two represent SPO and LPO. They have been discussed earlier. The third oscillatory mode represents the oscillation of the elevator about its hinge. The time period of this motion is about two seconds. This may sometimes lead to undesirable response of the airplane. The reason is as follows.

The response time of the pilot is about one second. Hence, the action of pilot, in response to an oscillatory motion with time period of two seconds, may reinforce the motion instead of correcting it. This may lead to instability. Proper damping of this mode is necessary. Section 10.6 of Ref.1.7 be consulted for further details.

Self study topic:

Handling characteristics of airplane. (These are discussed in Ref.1.1, at the end of chapter 4).