Chapter 8

Dynamic stability analysis – II – Longitudinal motion - 3

Lecture 30

Topics

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8.10 Equations of motion in state space or state variable form

The governing equations for the longitudinal motion (Eqs.7.85, 7.86 and 7.87) are ordinary differential equations with constant coefficients. When such equations are written as a system of first order differential equations, they are called state space or state variable equations and are written as:

$$\dot{\mathbf{X}} = \mathbf{A}.\ \mathbf{X} + \mathbf{B}.\ \boldsymbol{\eta} \qquad (8.34)$$

where, **X** is the state vector, **η** is the control vector and the matrices **A** and **B** contain stability derivatives. The steps for expressing the governing equations of longitudinal motions in state space variable form are as follows.

The equations of motion are reproduced below for ready reference.

$$\left(\frac{d}{dt} - X_u\right)\Delta u - X_w\ \Delta w + g\cos\theta_0\ \Delta\theta = X_{\delta e}\ \Delta\delta_e + X_{\delta T}\ \Delta\delta_T \qquad (7.85)$$

$$- Z_u\Delta u + \left[(1-Z_{\dot{w}})\frac{d}{dt} - Z_w\right]\Delta w - \left[(u_0 + Z_q)\frac{d}{dt} - g\sin\theta_0\right]\Delta\theta$$

$$= Z_{\delta e}\ \Delta\delta_e + Z_{\delta T}\ \Delta\delta_T \qquad (7.86)$$

$$- M_u\ \Delta u - \left(M_{\dot{w}}\frac{d}{dt} + M_w\right)\Delta w + \left(\frac{d^2}{dt^2} - M_q\frac{d}{dt}\right)\Delta\theta = M_{\delta e}\ \Delta\delta_e + M_{\delta T}\ \Delta\delta_T \qquad (7.87)$$

To bring out the essential ideas of the state variable form, a simpler set of equations is used. It is assumed that (a) θ_0 is zero i.e. the undisturbed flight is a level flight and (b) $Z_{\dot{w}}$ and Z_q are taken as zero.

The aforesaid set of equations (i.e.Eqs.7.85 to 7.87) now reduces to :

$$\left(\frac{d}{dt} - X_u\right)\Delta u - X_w\ \Delta w + g\ \Delta\theta = X_{\delta e}\ \Delta\delta_e + X_{\delta T}\ \Delta\delta_T \qquad (8.34a)$$

$$- Z_u\Delta u + \left[\frac{d}{dt} - Z_w\right]\Delta w - \left[\left(u_0 \frac{d}{dt}\right)\right]\Delta\theta = Z_{\delta e}\ \Delta\delta_e + Z_{\delta T}\ \Delta\delta_T \qquad (8.34b)$$

$$- M_u\ \Delta u - \left(M_{\dot{w}} \frac{d}{dt} + M_w\right)\Delta w + \left(\frac{d^2}{dt^2} - M_q \frac{d}{dt}\right)\Delta\theta = M_{\delta e}\ \Delta\delta_e + M_{\delta T}\ \Delta\delta_T \qquad (8.34c)$$

It is observed that the third equation in the above set (i.e. Eq.8.34 c) involves second derivative of $\Delta\theta$. But, the state variable form of the equations has only the first derivative of the dependent variables. To overcome this difficulty, $\left(\frac{d^2\theta}{dt^2}\right)$ is expressed as $\Delta\dot{q}$ and an additional equation, $\Delta\dot{\theta} = \Delta q$ is introduced. Then, the set of Eqs.(8.34a) to (8.34c), after some rearrangement, is expressed as:

$$\Delta\dot{u} = X_u\ \Delta u + X_w\ \Delta w + (0)\Delta q - g\Delta\theta + X_{\delta e}\Delta\delta_e + X_{\delta T}\ \Delta\delta_T \qquad (8.35)$$

$$\Delta\dot{w} = Z_u\ \Delta u + Z_w\Delta w + u_0\ \Delta q + (0)\Delta\theta + Z_{\delta e}\ \Delta\delta_e + Z_{\delta T}\ \Delta\delta_T \qquad (8.36)$$

$$\Delta\dot{q} = M_u\Delta u + M_{\dot{w}}\Delta\dot{w} + M_w\ \Delta w + M_q\ \Delta q + (0)\Delta\theta + M_{\delta e}\ \Delta\delta_e + M_{\delta t}\ \Delta\delta_T \qquad (8.37)$$

$$\Delta\dot{\theta} = \Delta q \qquad (8.38)$$

It is observed that Eq.(8.37) of this set involves the derivative of Δw on the right hand side. This again is not in accordance with the state variable representation (i.e. Eq.8.34). To overcome this difficulty, $\Delta\dot{w}$ in Eq.(8.37) is replaced by its expression as given by Eq.(8.36). Consequently, Eq.(8.37) is rewritten as :

$$\Delta\dot{q} = M_u\Delta u + M_{\dot{w}}\{Z_u\Delta u + Z_w\ \Delta w + u_0\ \Delta q + (0)\Delta\theta + Z_{\delta e}\ \Delta\delta_e + Z_{\delta t}\ \Delta\delta_T\} + M_w\Delta w + M_q\Delta q + M_{\delta e}\ \Delta\delta_e + M_{\delta t}\ \Delta\delta_T$$

$$\text{Or } \Delta\dot{q} = (M_u + M_{\dot{w}}Z_u)\Delta u + (M_w + M_{\dot{w}}Z_w)\Delta w + (M_q + M_{\dot{w}}u_0)\Delta q + (0)\Delta\theta + (M_{\delta e} + M_{\dot{w}}Z_{\delta e})\Delta\delta_e + (M_{\delta t} + M_{\dot{w}}Z_{\delta t})\Delta\delta_T \qquad (8.39)$$

Equations (8.35), (8.36), (8.39) and (8.38) are the alternate form of the governing equations for the longitudinal motion. These can be written in matrix form as:

$$\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{w}\\ \Delta\dot{q}\\ \Delta\dot{\theta}\end{bmatrix}=\begin{bmatrix}X_u & X_w & 0 & -g\\ Z_u & Z_w & u_0 & 0\\ M_u+M_{\dot{w}}Z_u & M_w+M_{\dot{w}}Z_w & M_q+M_{\dot{w}}u_0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta w\\ \Delta q\\ \Delta\theta\end{bmatrix}$$

$$+\begin{bmatrix}X_{\delta e} & X_{\delta T}\\ Z_{\delta_e} & Z_{\delta T}\\ M_{\delta e}+M_{\dot{w}}Z_{\delta e} & M_{\delta T}+M_{\dot{w}}\ Z_{\delta T}\end{bmatrix}\begin{bmatrix}\Delta\delta_e\\ \Delta\delta_T\end{bmatrix} \quad (8.40)$$

i.e., $\dot{\mathbf{X}} = \mathbf{A.X} + \mathbf{B.\eta}$ where

$$\dot{\mathbf{X}} = \begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{w}\\ \Delta\dot{q}\\ \Delta\dot{\theta}\end{bmatrix}, \mathbf{X} = \begin{bmatrix}\Delta u\\ \Delta w\\ \Delta q\\ \Delta\theta\end{bmatrix}, \mathbf{\eta} = \begin{bmatrix}\Delta\delta_e\\ \Delta\delta_T\end{bmatrix}, \quad (8.41)$$

$$\mathbf{A} = \begin{bmatrix}X_u & X_w & 0 & -g\\ Z_u & Z_w & u_0 & 0\\ M_u+M_{\dot{w}}Z_u & M_w+M_{\dot{w}}Z_w & M_q+M_{\dot{w}}u_0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix}X_{\delta e} & X_{\delta T}\\ Z_{\delta e} & Z_{\delta T}\\ M_{\delta e}+M_{\dot{w}}Z_{\delta e} & M_{\delta T}+M_{\dot{w}}Z_{\delta T}\end{bmatrix} \quad (8.42)$$

Remarks:

i) The quantities Δu, Δw, Δq and Δθ are called state variables and $\Delta\delta_e$ and $\Delta\delta_t$ are called control variables.

ii) When η = 0, Eq.(8.41) reduces to:

$$\dot{\mathbf{X}} = \mathbf{A.\ X.} \quad (8.43)$$

This set of equations has a solution:

$$\mathbf{X} = \mathbf{X}_r\, e^{\lambda_r t} \quad (8.44)$$

Substituting Eq.(8.44) in Eq.(8.34) gives:

$$(\lambda_r \mathbf{I} - \mathbf{A})\mathbf{X}_r = 0 \tag{8.45}$$

Where, the identity matrix **I** in this case is:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8.46}$$

For a non-trivial solution to exist:

$$|\lambda_r \mathbf{I} - \mathbf{A}| = 0. \tag{8.47}$$

Thus, λ_r's are the eigen values of matrix **A**. These are also the roots of the characteristics equation (Eq.8.9). Thus, the roots of the characteristic equation can be obtained by finding out the eigen values of matrix **A** using packages like Mathematica, Matlab etc.

i) For the general aviation airplane of example 8.1, the matrix **A** is :

$$\mathbf{A} = \begin{bmatrix} -0.045 & 0.036 & 0 & -9.80665 \\ -0.369 & -2.02 & 53.64 & 0 \\ 0.00612 & -0.1298 & -2.9862 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The roots of the equations using Matlab are:

$-2.5085 \pm i\ 2.5931$

$-0.01709 \pm i\ 0.2124$

which are almost the same as those obtained by the iterative procedure in example 8.2.

8.11 Approximations to modes of longitudinal motion

The response of an airplane to disturbances, discussed in section 8.9, shows that the changes in $\Delta\alpha$ or Δw take place in the first few moments of the disturbed motion. During this, the changes in Δu are negligible. In the subsequent motion, the angle of attack remains fairly constant and the velocity changes in a periodic manner. This change in velocity is accompanied by a change in the altitude of the airplane. This implies a gradual exchange between the kinetic energy and potential energy. These observations suggest that the analysis of SPO and LPO can be done by making simplifying assumptions. These simplified analyses are called approximations to SPO and LPO.

8.11.1 Approximation to SPO

From the aforesaid discussion the analysis of SPO is simplified by assuming that Δu is zero during this phase of motion. This results in the following simplifications.

a) In the set of governing equations (Eqs.7.85 to 7.87) the equation corresponding to X-force i.e. Eq (7.85) can be dropped.

b) From the other two equations the terms multiplied by Δu can be ignored.

c) Writing $\Delta\dot{\theta}$ as Δq, the following set of two equations is obtained.

$$\frac{d}{dt}\Delta w - Z_w \Delta w - u_0 \Delta q = 0 \tag{8.48}$$

$$\frac{dq}{dt} - (M_w + M_{\dot{w}} Z_w)\Delta w - (M_q + M_{\dot{w}} u_0)\Delta q = 0 \tag{8.49}$$

$$\text{or} \begin{bmatrix}\Delta\dot{w}\\ \Delta\dot{q}\end{bmatrix} = \begin{bmatrix} Z_w & u_0 \\ M_w + M_{\dot{w}} Z_w & M_q + M_{\dot{w}} u_0 \end{bmatrix}\begin{bmatrix}\Delta w\\ \Delta q\end{bmatrix} \tag{8.50}$$

Now, Δw can be replaced by $u_0 \Delta\alpha$.

$$\text{Further,}\quad M_\alpha = \frac{1}{I_{yy}}\frac{\partial M}{\partial\alpha} = \frac{1}{I_{yy}}\frac{\partial M}{\partial(\Delta w/u_0)} = \frac{u_0}{I_{yy}}\frac{\partial M}{\partial w} = u_0 M_w \tag{8.51}$$

Similarly, $Z_\alpha = u_0 Z_w$ and $M_{\dot{\alpha}} = u_0 M_{\dot{w}}$. Substituting these, the simplified equations for SPO become:

$$\begin{bmatrix}\Delta\dot{\alpha}\\ \Delta\dot{q}\end{bmatrix} = \begin{bmatrix} \frac{Z_\alpha}{u_o} & 1 \\ (M_\alpha + M_{\dot{\alpha}}\frac{Z_\alpha}{u_0}) & M_q + M_{\dot{\alpha}} \end{bmatrix}\begin{bmatrix}\Delta\alpha\\ \Delta q\end{bmatrix} \tag{8.52}$$

The characteristic equation for this mode can be obtained from $|\lambda\mathbf{I} - \mathbf{A}| = 0$ or

$$\begin{vmatrix} \lambda - \frac{Z_\alpha}{u_0} & -1 \\ -M_\alpha - M_{\dot{\alpha}}\frac{Z_\alpha}{u_0} & \lambda - (M_q + M_{\dot{\alpha}}) \end{vmatrix} = 0 \tag{8.53}$$

Expanding this, gives:

$$\lambda^2 - (M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0})\lambda + (M_q \frac{Z_\alpha}{u_0} - M_\alpha) = 0 \qquad (8.54)$$

Solving Eq.(8.54) yields:

$$\lambda_{SPO} = \frac{1}{2}[\{M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0}\} \pm \{(M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0})^2 - 4(M_q \frac{Z_\alpha}{u_0} - M_\alpha)\}^{1/2}] \qquad (8.55)$$

Remark:

Substituting in Eq.(8.55), the values of stability derivatives for the airplane in example 8.1, gives :

λ_{SPO} = - 2.503 ± i 2.594 whereas the exact roots for SPO are -2.508 ± i 2.577.

8.11.2 Approximation to LPO

Here, it is assumed that $\Delta\alpha$ or $\Delta w/u_0$ is small and changes occur only in Δu and $\Delta\theta$. Changes in $d\theta/dt$ are also slow. Hence, the moment equation i.e. Eq.(7.87) is ignored. Retaining only Δu and $\Delta\theta$ in the remaining equations of motion, gives:

$$(\frac{d}{dt} - X_u)\Delta u + g\,\Delta\theta = 0 \qquad (8.56)$$

$$- Z_u\,\Delta u - u_0 \frac{d}{dt}\Delta\theta = 0 \qquad (8.57)$$

$$\text{Or} \begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & -g \\ \frac{-Z_u}{u_0} & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\theta \end{bmatrix} \qquad (8.58)$$

The characteristic equation for this mode is obtained from $|\lambda\mathbf{I} - \mathbf{A}| = 0$ or

$$\begin{vmatrix} \lambda - X_u & g \\ \frac{Z_u}{u_0} & \lambda \end{vmatrix} = 0$$

$$\text{Or } \lambda^2 - X_u\lambda + \frac{Z_u g}{u_0} = 0 \qquad (8.59)$$

$$\lambda_{LPO} = \frac{1}{2}[X_u \pm \{X_u^2 - 4\frac{Z_u g}{u_0}\}^{1/2}] \qquad (8.60)$$

Remarks:

i) Substituting in Eq.(8.60) the values of the stability derivatives for the airplane in example 8.1, gives :

λ_{LP0} = - 0.0225 ± i 0.257 as compared to the exact value of - 0.01715 ± i 0.2135

ii) From Eq.(8.60) the damping of LPO is ½ X_u.

It may be recalled that

$$X_u = -\frac{2\,Q\,S\,C_D}{m\,u_0} \qquad (8.61)$$

Hence, damping of phugoid depends on C_D. Thus, a streamlined airplane has lower damping.

iii) The frequency of phugoid depends on

$$\frac{1}{2}\{X_u^2 - 4\,\frac{Z_u\,g}{u_0}\}^{1/2}$$

Since, X_u^2 is much smaller than ($4Z_u\,g / u_0$) the frequency (ω_{LPO}) is roughly equal to $(-Z_u\,g/u_0)^{1/2}$

Recall that:

$$Z_u = -\frac{2QSC_L}{mu_0} = -\frac{2QS}{mu_0}\times\frac{2mg}{QS} = -\frac{2g}{u_0} \qquad (8.62)$$

Hence, the frequency of phugoid is given approximately by:

$$\omega_{LP0} \approx \sqrt{2}\,\frac{g}{u_0} \qquad (8.63)$$

Thus, the period of phugoid is proportional to flight speed (u_o) or the period is longer at higher flight speeds. This result is the same as arrived at in subsection 8.9.1.