Chapter 8

Dynamic stability analysis – II – Longitudinal motion - 2

Lecture 29

Topics

8.5 Iterative solution of the characteristic equation

The characteristic equation for the stick-fixed longitudinal motion, given above, is a fourth degree polynomial. It may be noted that the exact solutions for polynomial equations are available only up to polynomials of degree three. Hence, in the present case, with fourth degree polynomial, an iterative procedure is adopted to obtain the solution. Further, the iterative technique, to be used for a fourth degree polynomial, depends on the relative magnitudes of the coefficients of the terms in the polynomial. The characteristic equation of the longitudinal motion generally has the following features.

(a) The coefficient of λ^4 is unity.

(b) The coefficients of λ^3 and λ^2 are much larger than the coefficient of the λ and the constant term (see values of B, C, D and E in Eq.8.18).

Reference1.5 Appendix 4, gives the following iterative procedure for solving polynomial like in Eq.(8.18).

Let, $f(\lambda) = \lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E$ (8.19)

Now, f(λ) can be expressed as a product of two quadratics i.e.:

$$f(\lambda) = (\lambda^2 + b\lambda + c)(\lambda^2 + \gamma\lambda + \delta) \quad (8.20)$$

Expanding R.H.S. of Eq.(8.20) and comparing coefficients of terms in Eqs. (8.19) and (8.20) gives the following equations.

$$\begin{aligned} b + \gamma &= B \\ c + b\gamma + \delta &= C \\ b\delta + c\gamma &= D \\ c\delta &= E \end{aligned} \quad (8.21)$$

Since, the coefficients D and E are generally much smaller than 'B' and 'C' , the quantities γ and δ in Eq.(8.21) are much smaller than 'b' and 'c' . Hence, the first approximations of b,c, δ and γ, denoted by b_1, c_2, δ_1 and γ_1 are written as :

$$\begin{aligned} b_1 &\approx B \\ c_1 &\approx C \\ \delta_1 &= \frac{E}{c} \approx \frac{E}{C} \\ \gamma_1 &= \frac{cD - bE}{c^2} \approx \frac{CD - BE}{C^2} \end{aligned} \quad (8.22)$$

Hence, $f(\lambda) \approx (\lambda^2 + B\lambda + C)\left(\lambda^2 + \frac{CD - BE}{C^2}\lambda + \frac{E}{C}\right) \approx 0$ (8.23)

The roots of the two quadratics in Eq.(8.23) are obtained.

For the second approximation, b_2, c_2 , δ_2, and γ_2 are expressed as:

$$\begin{aligned} b_2 &\approx B - \gamma_1 \\ c_2 &\approx C - b_1\gamma_1 - \delta_1 \\ \gamma_2 &\approx \frac{c_2 D - b_2 E}{c_2^2} \\ \delta_2 &= \frac{E}{c_2} \end{aligned} \quad (8.24)$$

The roots of the quadratics obtained using b_2 , c_2, γ_2 and δ_2 are also worked out.The procedure is continued till the roots obtained in the two consecutive approximations do not change significantly in their values.

Example 8.2

An application of the iterative procedure for the characteristic equation for the case considered in example 8.1 is described below.

From Eq.(8.18): A = 1, B = 5.05, C = 13.15, D = 0.6735 and E = 0.593.

From the set of equations in Eq.(8.22) the first approximation is :

$b_1 = 5.05$, $c_1 = 13.15$, $\delta_1 = (0.593/13.15) = 0.0451$, and

$\gamma_1 = (13.15 \times 0.6735 - 5.05 \times 0.593) / 13.15^2 = 0.0339$

Thus, $f(\lambda) \approx (\lambda^2+5.05\lambda+13.15)\,(\lambda^2 + 0.0339\lambda + 0.0451)$

The roots of the two quadratics are:

$\lambda_{1,2} = -2.525 \pm i\ 2.602$, $\lambda_{3,4} = -0.01695 \pm i\ 0.2117$

From the set of equations in Eq.(8.24) the second approximation is:

$b_2 = B - \gamma_1 = 5.05 – 0.0339 = 5.0161$.

Similarly, c_2=12.934, $\gamma_2 = 0.0343$ and $\delta_2 = 0.0458$.

The roots of the quadratic obtained using b_2, c_2, γ_2 and δ_2 are:

$\lambda_{1,2} = -2.508 \pm i\ 2.578$, $\lambda_{3,4} = -0.01715 \pm i\ 0.213$

Carrying out the iteration once more, the roots after the third approximation are:

$\lambda_{1,2} = -2.508 \pm i\ 2.577$,

$\lambda_{3,4} = -0.01715 \pm i\ 0.2135$.

The values of the roots do not seem to change significantly from the second to the third iteration and the iterative procedure can be stopped.

Remarks:

i) In section 8.10 the equations of motion are expressed in state space variable form and then the roots of the characteristics equation are obtained by using commercially available computational packages like Matlab.

ii) It is observed that the four roots of the characteristic equation for the given flight condition, consist of two pairs of complex roots. The real parts of both the roots are negative and hence the airplane has dynamic stability for the given flight conditions and configuration. A discussion on the modes of longitudinal motion is given in section 8.9.

8.6 Routh's Criteria

Presently, the roots of the stability quartic are obtained by the iterative procedure described above or by using packages like Matlab. However, earlier the tendency was to look for elegant analytical / approximate solutions. Routh's criteria is a method which indicates whether a system is stable without solving the characteristic equation. The criteria is presented without giving the mathematical proof.

A quartic $A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$ will have roots indicating stability i.e. real roots negative and complex roots with negative real part when A>0 and the functions T_1,T_2, T_3 and T_4, given below, are positive.

$$T_1 = B$$

$$T_2 = \begin{vmatrix} B & A \\ D & C \end{vmatrix} = BC - AD$$

$$T_3 = \begin{vmatrix} B & A & 0 \\ D & C & B \\ 0 & E & D \end{vmatrix} = DT_2 - B^2E = BCD - B^2E - AD^2 \tag{8.25}$$

$$T_4 = \begin{vmatrix} B & A & 0 & 0 \\ D & C & B & A \\ 0 & E & D & C \\ 0 & 0 & 0 & E \end{vmatrix} = ET_3$$

In the case of longitudinal stability quartic with A = 1, the criteria simplify to:

$B > 0$; $D > 0$; $E > 0$ and

$$R = T_3 = BCD - B^2E - AD^2 > 0 \tag{8.26}$$

The term 'R' is called Routh's discriminant.

The reader can verify that for the stability quartic given by Eq.(8.18), the value of R is positive.

8.7 Damping and rate of divergence when roots are real

As mentioned earlier, when a root is real and non-zero, a negative root indicates subsidence and a positive root indicates divergence. Larger the magnitude of the negative root, faster will the system return to the undisturbed position. This is clear from Eq.(8.15), which shows that the response of the system corresponding to the root λ_1 is $\rho_{11}\,e^{\lambda_1 t}$. At t = 0, the amplitude of the response is ρ_{11}. Further, when λ_1 is negative, the term $e^{\lambda_1 t}$ indicates that the amplitude would decrease exponentially with time (Fig 8.1b). The time when the amplitude decreases to half of its value at t = 0, is a measure of the damping. This time is denoted by $t_{1/2}$. This quantity ($t_{1/2}$) is obtained from the following equation.

For the sake of generality the root is denoted by λ instead of λ_1.

$$e^{t_{1/2}\,\lambda} = \frac{1}{2}$$

$$\text{Or} \quad t_{1/2} = (\ln 2) / |\lambda| = 0.693 / |\lambda| \qquad (8.27)$$

When the root is positive, the amplitude increases exponentially with time (Fig 8.1a). The time when the amplitude is twice the value at t = 0, is a measure of divergence. This time is denoted by t_2. This quantity (t_2) is obtained from the following equation.

$$e^{t_2\,\lambda} = 2; \text{ ; Note } \lambda \text{ is positive}$$

$$\text{Or} \quad t_2 = (\ln 2) / \lambda = 0.693 / \lambda \qquad (8.28)$$

8.8 Damping, rate of divergence, period of oscillation and number of cycles for half amplitude when the roots constitute a complex pair

A complex root is usually written as:

$$\lambda = \eta \pm i\omega$$

When η is negative, the response is a damped oscillation. The damping is characterized by the time when the quantities $e^{\eta t}$ becomes half. This time is denoted by $t_{\frac{1}{2}}$. Consequently ,

$$e^{\eta\, t_{\frac{1}{2}}} = 0.5$$

$$\text{Or} \quad t_{1/2} = (\ln 2) / |\eta| = 0.693 / |\eta| \qquad (8.29)$$

When η is positive, the response is a divergent oscillation. The time when the term $e^{\eta t}$ equals two is a measure of the rate of divergence. This time is denoted by t_2. It is easy to show that :

$$t_2 = (\ln 2)/\eta = 0.693/\eta \qquad (8.30)$$

The time period of the oscillation (P) is given by

$$P = 2\pi / \omega \qquad (8.31)$$

When η is negative, the number of cycles from t = 0 to $t_{1/2}$ is denoted by $N_{1/2}$ and equals:

$$N_{1/2} = t_{1/2} / P \qquad (8.32)$$

Similarly, when η is positive, the number of cycles from t = 0 to t_2 is denoted by N_2 and equals:

$$N_2 = t_2 / P \qquad (8.33)$$

Example 8.3

Applying the above formulae, the quantities $t_{1/2}$, P, $N_{1/2}$ corresponding to the two roots obtained in example 8.2 are:

a) $\lambda_{1,2}$ = -2.508 ± i 2.578:

$t_{1/2}$ = 0.693/2.508 = 0.276 s,

P = 2π /2.578 = 2.436 s, $N_{1/2}$ = 0.276 / 2.436 = 0.113 cycles

b) $\lambda_{3,4}$ = - 0.01715 ± i 0.2135:

$t_{1/2}$ = 0.693 / 0.01715 = 40.4s

P = 2π / 0.2135 = 29.4 s, $N_{1/2}$ = 40.4 / 29.4 = 1.37 cycles

8.9 Modes of longitudinal motion – short period oscillation (SPO) and long period oscillation (LPO) / phugoid

The motions represented by the different roots of the characteristic equation are called the corresponding modes. For the longitudinal motion the characteristic equation generally has two complex roots. One of them has a short time period and is heavily damped (refer to examples 8.2 and 8.3). This mode is called short period oscillation (SPO). The other mode has a long time period and low damping (refer to examples 8.2 and 8.3).This mode is called long period oscillation(LPO). Because of low damping it takes a long time to subside or die down. However, when the period is long, the pilot has enough time to take corrective action to restore equilibrium. This mode is also called phugoid. To appreciate the features of these two modes, the response of Navion to a disturbance, as given in Reference 1.12, chapter 6 is presented in Figs.8.3a & b and 8.4 a & b.The disturbance consists of $\Delta\alpha = 5^0$ and $\Delta u = 0.1u_0 = 17.6$ ft s $^{-1}$ or 5.364 ms^{-1}.

Figures 8.3 a and b show the changes, with time, in the angle of attack (Δα) and pitch rate (q). Figures 8.4 a and b show the changes, with time, in the perturbation velocity (Δu) and the pitch angle (Δθ). Note that Fig.8.3 shows the initial response to disturbance i.e. up to t = 20 s whereas Fig.8.4 shows the response up to t = 200 s. It may also be pointed out that the solid curves in Figs.8.3 and 8.4 represent the response of the airplane when the full set of equations, (namely Eqs.(7.85), (7.86) and (7.87)) is used. This set of equations is referred to as fourth order system. The dotted lines represent approximate solutions to short period oscillation and phugoid motions. These approximations are dealt with in section 8.11.

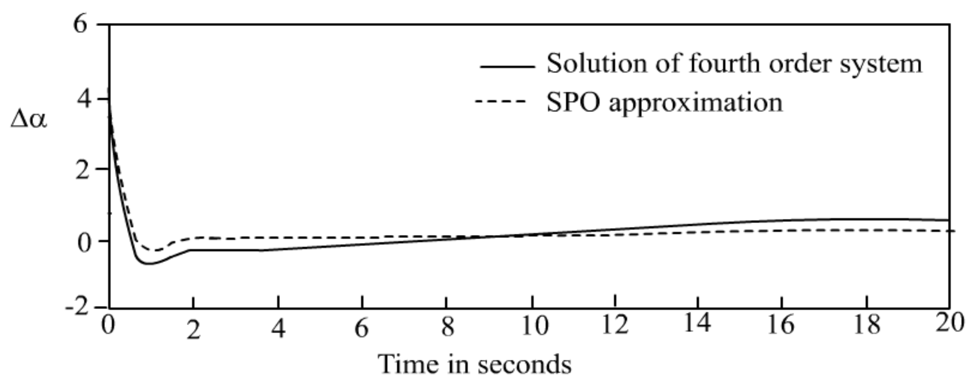

Fig.8.3a Response of general aviation airplane to disturbance in Δα and Δu - change in Δα (Adapted from Ref.1.12, chapter 6 with permission from American Institute of Aeronautics and Astronautics, Inc.)

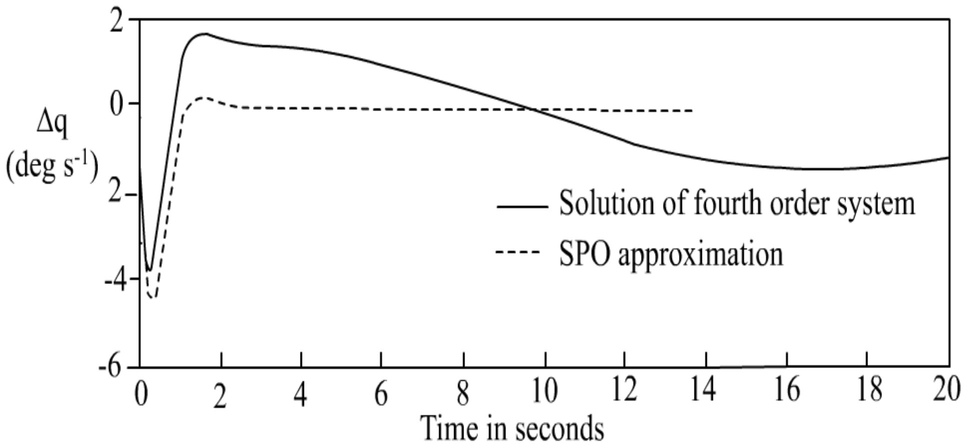

Fig.8.3b Response of general aviation airplane to disturbance in Δα and Δu - change in Δq (Adapted from Ref.1.12, chapter 6 with permission from American Institute of Aeronautics and Astronautics, Inc.)

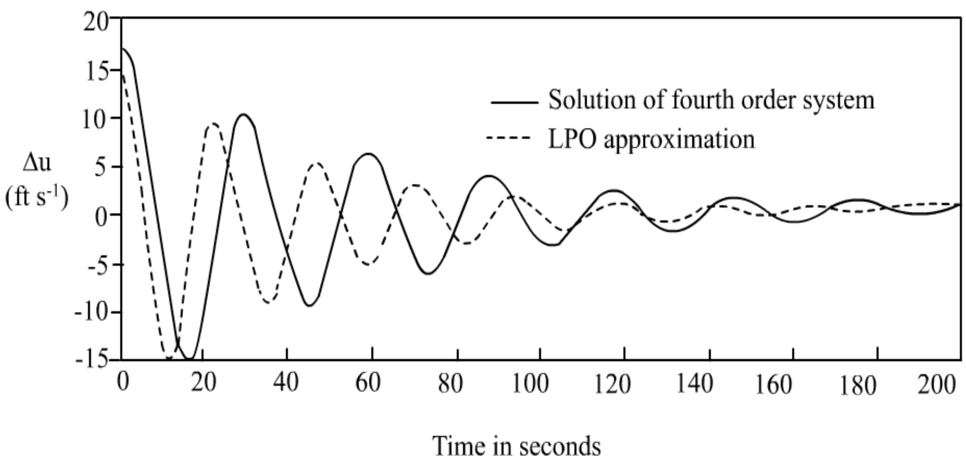

Fig.8.4a Response of general aviation airplane to disturbance in Δα and Δu – change in Δu (Adapted from Ref.1.12, chapter 6 with permission from American Institute of Aeronautics and Astronautics, Inc.)

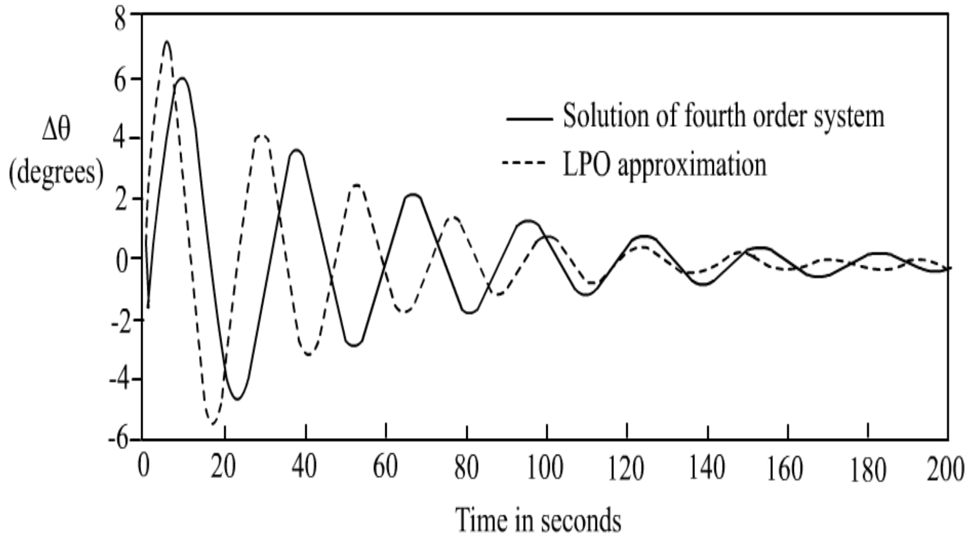

Fig.8.4b Response of general aviation airplane to disturbance in Δα and Δu - change in Δθ (Adapted from Ref.1.12, chapter 6 with permission from American Institute of Aeronautics and Astronautics, Inc.)

From Fig.8.3a it is observed that rapid changes in angle of attack and pitch rate take place in the first few seconds after the disturbance. Figure 8.4a shows that the changes in velocity are negligible in the first few seconds and extend over a long period of time. Keeping these observations in mind and noting the values of the roots of the characteristics equation the following observations are made.

a) The short period oscillation with heavy damping influences the motion during the first few seconds. During this period, the angle of attack and pitch rate change rapidly whereas the velocity remains approximately constant. However, within this short time the angle of attack is nearly restored to its initial undisturbed value and remains so thereafter.

b) The long period oscillation persists after the SPO has died down and influences the changes in velocity and pitch angle in a periodic manner. The angle of attack remains almost constant. Figures 8.4 a and b also show that after the initial transient motion due to SPO dies out, the motion is goverened by the LPO whose period is about 29 s (see example 8.3).The amplitude of the velocity fluctuation is nearly half of the initial amplitude after about one and half cycles (note that $N_{1/2}$ in example 8.3, is 1.43 cycles). However, it takes long time for the motion to die out.

8.9.1 Phugoid as slow interchange of kinetic energy and potential energy

The following aspects of Phugoid are observed.

(a)As the pitch angle goes through a cycle (Fig. 8.4 b), while the angle of attack remains nearly constant implies that the altitude of the airplane also changes in a periodic manner (Fig 8.5).

(b)The damping of the phugoid is very light and the flight speed changes periodically.

(c) Items (a) and (b) suggest that the motion, during one cycle, can be considered as an exchange between potential energy and kinetic energy of the airplane.The total energy (i.e. sum of potential and kinetic energies) remains nearly constant during the cycle.

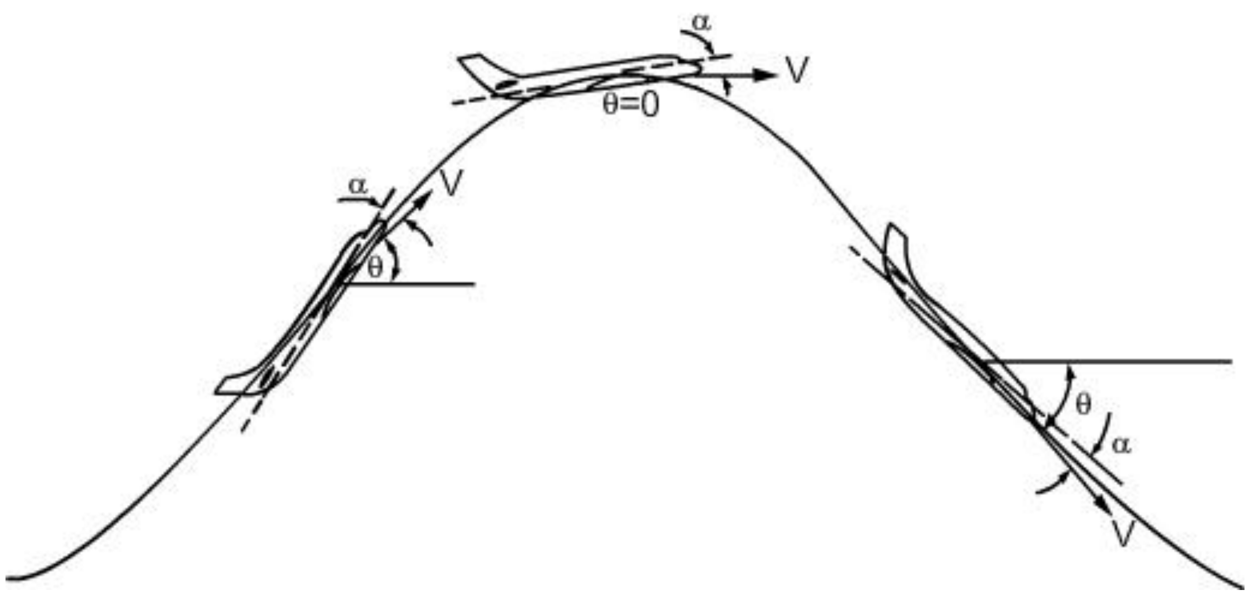

Fig.8.5 Schematic of the motion of airplane in LPO

Fig.8.5 shows half of the cycle during a Phugoid. It is observed that as the airplane climbs up, its potential energy increases. At the same time, the flight speed decreases and the airplane loses kinetic energy. At the crest, the flight speed is minimum. Conversely, when the altitude decreases the flight speed increases and is maximum at the trough.

In the 1979 edition of Ref.1.9 chapter an expression for the frequency of phugoid is derived based on this exchange between potential energy and kinetic energy and neglecting damping. The derivation is presented below, in a different manner. The basis is as follows.

At an instant, during the cycle, the net force in the vertical direction is:

$$Z = W - L = W - \frac{1}{2}\rho V^2 S C_L$$

$$= W - \frac{1}{2}\rho (u_0 + \Delta u)^2 S C_L$$

$$= W - \frac{1}{2}\rho (u_0^2 + 2u_0 \Delta u + \Delta u^2) S C_L$$

Ignoring Δu^2 as small compared to the other terms in the bracket yields:

$$Z = W - \frac{1}{2}\rho (u_0^2 + 2u_0 \Delta u) S C_L \qquad (8.33a)$$

Noting that, (a) $W = \frac{1}{2}\rho u_0^2 S C_L$ and (b) C_L is constant when α is constant,

$$Z = -\rho u_0 S C_L \Delta u \qquad (8.33b)$$

Let $\ddot{z}$ denotes the acceleration in the vertical direction. Then:

$$Z = m\ddot{z} = -\rho u_0 S C_L \Delta u \qquad (8.33c)$$

Let, the solution of Eq.(8.33c) be:

$$\Delta u = \Delta u_{max} \sin\omega t \qquad (8.33d)$$

Substituting for Δu in Eq.(8.33c) yields:

$$m\ddot{z} = -\rho u_0 S C_L \Delta u_{max} \sin\omega t \qquad (8.33e)$$

Integrating Eq.(8.33e) gives:

$$z = \frac{\rho u_0 S C_L \Delta u_{max}}{m\omega^2} \sin\omega t + C_1 t + C_2 \qquad (8.33f)$$

Noting that (a) z = 0 at t = 0 and (b) $\dot{z} = 0$ at $t = \frac{\pi}{2\omega}$, gives $C_1 = C_2 = 0$

Or $$z = \frac{\rho\, u_0\, S\, C_L\, \Delta u_{max}}{m\, \omega^2} \qquad (8.33g)$$

Now, the altitude changes as z changes and the potential energy changes with it. The maximum change in potential energy $(\Delta PE)_{max}$ is mg (z_{max} – z_{min}). From Eq.(8.33g)

$$(\Delta PE)_{max} = \frac{2\, g\, \rho\, u_0\, S\, C_L\, \Delta u_{max}}{\omega^2} \qquad (8.33h)$$

The maximum change in the kinetic energy $(\Delta KE)_{max}$ is:

$$(\Delta KE)_{max} = \frac{1}{2}\, m\, (u_0 + \Delta u_{max})^2 - \frac{1}{2} m (u_0 - \Delta u_{max})^2 = 2 m u_0\, \Delta u_{max} \qquad (8.33i)$$

Since, the phugoid is approximated as an exchange between PE and KE,

$$(\Delta PE)_{max} = (\Delta KE)_{max}$$

Or $$\frac{2 g \rho u_0 S C_L \Delta u_{max}}{\omega^2} = 2 m u_0\, \Delta u_{max} \qquad (8.33j)$$

Substituting $C_L = \frac{2mg}{\rho u_0^2 S}$ in Eq.(8.33j) yields:

$$\omega = \sqrt{2}\, \frac{g}{u_0} \text{ rad/s} \qquad (8.33\ k)$$

This interesting result shows that the frequency of phugoid is inversely proportional to u_0 or the time period of the oscillation is proportional to u_0. The result is independent of the airplane characteristics because the damping (which depends on C_D) has been ignored in the analysis.

In subsection 8.11.2 the same expression for ω is obtained by simplifying the equations of motion.

Remark: The website: www.youtube.com has many videos on Phugoid. One of them (PH-Lab (3/7)) indicates phugoid motion as the movement of horizon when photographed from inside the airplane. See also videos on short period oscillation.