# Chapter 7

# Dynamic stability analysis – I – Equations of motion and estimation of stability derivatives - 6

## Lecture 27

## Topics



### 7.17 Derivatives due to change of $'\beta'$

These derivatives include $C_{y\beta}$ , $C_{n\beta}$ and $C'_{l\beta}$

### 7.17.1 $C_{y\beta}$

The major contribution to this quantity is from the vertical tail. From Eq.(5.13):

$$(Y)_{v.tail} = -\frac{1}{2}\rho V_v^2 S\ C_{L\alpha v}\ \alpha_v \tag{7.146}$$

Noting that $\alpha_v = \beta+\sigma$, $\eta_v = (½ \rho V_v^2)/ (½ \rho V^2)$and $C_y = Y / (½ \rho V^2 S)$ , Eq.(7.146) gives:

$$C_{y\beta} = -\eta_v \frac{S_v}{S} C_{L\alpha v}\ (1+\frac{d\sigma}{d\beta}) \tag{7.147}$$

It may be added that $Y_\beta = \frac{QSC_{y\beta}}{m}$ (7.148)

**Remark:**

Reference 1.8b, chapter 7 mentions that wing and fuselage also have small contributions to $C_{y\beta}$ (see also Appendix 'C').

### 7.17.2 $C_{n\beta}$

From Eq.(5.21):

$$C_{n\beta} = (C_{n\beta})_w + (C_{n\beta})_{f,n,p} + V_V\,\eta_v\,C_{L\alpha v}(1+\frac{d\sigma}{d\beta}) \tag{7.149}$$

Details regarding estimation of various terms in Eq.(7.149) are given in Sections 5.3 to 5.6.

Note that $N_\beta = \frac{QSb\,C_{n\beta}}{I_{zz}}$ (7.150)

### 7.17.3 $C'_{l\beta}$

From Eq.(6.2):

$$C'_{l\beta} = (C'_{l\beta})_w + (C'_{l\beta})_{f,n,p} + (C'_{l\beta})_{vt} \tag{7.151}$$

Details regarding estimation of different terms in Eq.(7.151) are given in section 6.5 to 6.8.

Note that $L'_\beta = \frac{QSb\,C'_{l\beta}}{I_{xx}}$ (7.151a)

## 7.18 Derivatives due to change of 'p'

These derivatives include $C_{yp}$, $C_{np}$ and $C'_{lp}$

### 7.18.1 $C_{yp}$

Consider an airplane rolling with angular velocity 'p'. A component of the airplane which is at a distance 'z' from the x-axis would be subject to a linear velocity 'pz'. This would induce a sideslip angle $\beta = pz / u_0$. This sideslip angle would result in a side force. Since, the sideslip angle depends on the distance 'z' from the x-axis, the contribution to $C_{yp}$ is mainly due to the vertical tail. When subjected to this sideslip, the vertical tail would produce a side force. The vertical location of this side force would depend on $Z_v$, $l_v$ and the angle of attack of the airplane in the undisturbed flight; $Z_v$ is the distance of the a.c. of vertical tail above of c.g.

Reference 1.8b, chapter 8 gives the following expression:

$$C_{yp} \approx (C_{yp})_{v.tail} = 2\frac{(Z_v \cos\alpha - l_v \sin\alpha)}{b} C_{y\beta v} \qquad (7.152)$$

Note that $Y_p = \dfrac{QS\,bC_{yp}}{2mu_0}$ (7.153)

**Remark:** $C_{yp}$ is generally small and neglected.

### 7.18.2 $C_{np}$

The main contribution to this quantity is from the wing in the form of adverse yaw. As explained in section 5.8.1:

$$(C_n)_{adverse\,yaw} = -\frac{C_L}{8}\frac{pb}{2u_0} \qquad (7.154)$$

Hence, $C_{np} = \dfrac{dC_n}{d(\frac{pb}{2u_0})} = -\dfrac{C_L}{8}$ (7.155)

Note : $N_p = \dfrac{QS\,b^2 C_{np}}{2I_{zz}u_0}$ (7.155a)

**Remark:**

Reference 1.8b gives a procedure which includes contributions due to vertical tail (see also Appendix 'C').

### 7.18.3 $C'_{lp}$

Main contribution $C'_{lp}$ to this quantity is from the damping due to wing. From Eq.(6.22):

$$(C'_l)_{damp} = \frac{2C_{L\alpha w}\, p}{u_0 S b} \int_0^{b/2} c y^2\, dy \tag{7.156}$$

Hence,

$$C'_{lp} = \frac{\partial C'_l}{\partial\left(\frac{pb}{2u_0}\right)} = \frac{4\, C_{L\alpha w}}{Sb^2} \int_0^{b/2} c\, y^2\, dy \tag{7.157}$$

Note that: $$L'_p = \frac{QS\, b^2 C'_{lp}}{2I_{xx} u_0} \tag{7.158}$$

**Remarks:**

i) The horizontal tail and vertical tail also make minor contributions to $C'_{lp}$ (see Appendix 'C').

ii) Reference 1.1, chapter 3 gives the following formula for $C'_{lp}$ in terms of $C_{L\alpha w}$ and geometrical parameters of the wing.

$$C'_{lp} = -\frac{C_{L\alpha}}{12} \frac{1+3\lambda}{1+\lambda} \tag{7.158a}$$

## 7.19 Derivatives due to change of 'r'

These derivatives include $C_{yr}$, $C_{nr}$ and $C'_{lr}$

### 7.19.1 $C_{yr}$

When an airplane has an angular velocity in yaw (r) , then a component of the airplane at a distance 'x' from the c.g. experiences a side word velocity of 'rx'. This results in side slip of ($rx/u_0$) leading to side force and consequently $C_{yr}$. The major contribution to $C_{yr}$ is due to the vertical tail. It is evident that the vertical tail has a side ward velocity of r $l_v$ and side slip angle $\Delta\beta$ of ( $-rl_v / u_0$).

This would produce a side force:

$$Y = -\, C_{L\alpha v}\, \Delta\beta\, Q_v\, S_v \tag{7.159}$$

Hence, $$C_y = \frac{C_{L\alpha v} \left(\frac{l_v\, r}{u_0}\right) Q_v\, S_v}{QS} = C_{L\alpha v} \left(\frac{l_v\, r}{u_0}\right) \eta_v \frac{S_v}{S} \tag{7.160}$$

Let, $C_{yr} = \dfrac{\partial C_y}{\partial(\dfrac{rb}{2u_0})}$, Then, $C_{yr} = 2 C_{L\alpha v}\, \eta_v \dfrac{S_v}{S} \dfrac{l_v}{b}$ (7.161)

Noting that, $(C_{y\beta})_{v.tail} \approx - C_{L\alpha v}\ \eta_v\ \dfrac{S_v}{S}$,

$$C_{yr} = -2(C_{y\beta})_{v.tail} \frac{l_v}{b} \quad (7.162)$$

Note that: $Y_r = \dfrac{Q\,S\,b\,C_{yr}}{2\,mu_0}$ (7.163)

**Remark:**

$Y_r$ is generally small and in Eq.(7.88) it appears as $(u_0 - Y_r)$. Hence $Y_r$ is often neglected.

### 7.19.2 $C_{nr}$

As mentioned above when an airplane has an angular velocity ‘r’ the vertical tail experiences a sideslip angle $\Delta\beta = (-l_v\, r / u_0)$ . The side force would also produce a yawing moment given by:

$$N = C_{L\alpha v}\, \Delta\beta\, Q_v S_v l_v = -C_{L\alpha v} \left(\frac{l_v\, r}{u_0}\right) Q_v S_v l_v \quad (7.164)$$

$$\text{Or } C_n = \frac{N}{qSb} = -\, C_{L\alpha v}\ \left(\frac{l_v\ r}{u_0}\right) \eta_v V_v \quad (7.165)$$

$$\text{Let, } C_{nr} = \frac{\partial C_n}{\partial(\dfrac{rb}{2u_0})}, \quad (7.166)$$

$$\text{Then, } C_{nr} = -2 C_{L\alpha v}\, \eta_v\, V_v\, \frac{l_v}{b} \quad (7.167)$$

$$\text{Note that :} \quad N_r = \frac{QSb^2\, C_{nr}}{2 I_{zz} u_0} \quad (7.168)$$

**Remark:**

When the airplane has a rate of yaw, the two wing halves experience different velocities (refer to the explanation of $C'_{lr}$ in the next subsection) and hence, experience different drags. This would cause a small contribution from wing to $C_{nr}$ (see appendix 'C').

### 7.19.3 $C'_{lr}$

Consider an airplane experiencing a positive rate of yaw i.e. right wing back. Then, a section of the right wing at a distance 'y' from the x-axis would experience a velocity 'ry' or relative wind of ($u_o$-ry). Consequently, the dynamic pressure is lower on the right wing. For the same reason, the dynamic pressure is higher on the left wing. The lifts on the two wing halves are different, thus producing a positive rolling moment. Strip theory (as explanied in subsection 6.10.2) can be used to obtain an approximate estimate of $C'_l$ and $C'_{lr}$. In addition to this the side force on the vertical tail also contributes to $C'_{lr}$. Ref.1.1, chapter 3 gives the following approximate formula:

$$C'_{lr} = \frac{C_L}{4} - 2\frac{l_v}{b}\frac{Z_v}{b}C_{y\beta tail} \tag{7.168a}$$

Reference 1.8b gives an improved estimates. See also Appendix 'C'.

Note that: $$L'_r = \frac{QSb^2 C'_{lr}}{2I_{xx}u_0} \tag{7.169}$$

## 7.20 Lateral control derivatives:

Expressions for the derivatives $C_{n\delta r}$ and $C'_{l\delta a}$ are given in Eq.(5.31a) and (6.27) respectively. The cross derivatives viz. $C_{n\delta a}$ and $C'_{l\delta r}$ can be obtained from Ref. 1.1 and 1.12.

## 7.21 Summary of lateral stability derivatives

The expressions for the lateral stability derivatives are summarized in Table 7.7; Table 7.6 presents the list of symbols.

| | | |
|---|---|---|
| $Y_\beta = \frac{QSC_{y\beta}}{m}$ $(m/s^2)$ | $Y_p = \frac{QSC_{yp}}{2mu_0}$ $(m/s)$ | $Y_r = \frac{QSC_{yr}}{2mu_0}$ $(m/s)$ |
| $N_\beta = \frac{QSbC_{n\beta}}{I_{zz}} (s^{-2})$ | $N_p = \frac{QSb^2C_{np}}{2I_{zz}u_0}$ $(s^{-1})$ | $N_r = \frac{QSb^2C_{nr}}{2I_{zz}u_0}$ $(s^{-1})$ |
| $L'_\beta = \frac{QSbC'_{l\beta}}{I_{xx}} (s^{-2})$ | $L'_p = \frac{QSb^2C'_{lp}}{2I_{xx}u_0}$ $(s^{-1})$ | $L'_r = \frac{QSb^2C'_{lr}}{2I_{xx}u_0}$ $(s^{-1})$ |
| $Y_{\delta a} = \frac{QSC_{y\delta a}}{m}$ $(m/s^2)$ | $Y_{\delta r} = \frac{QSC_{y\delta r}}{m}$ $(m/s^2)$ | |
| $N_{\delta a} = \frac{QSbC_{n\delta a}}{I_{zz}} (s^{-2})$ | $N_{\delta r} = \frac{QSbC_{n\delta r}}{I_{zz}}$ $(s^{-2})$ | |
| $L'_{\delta a} = \frac{QSbC'_{l\delta a}}{I_{xx}} (s^{-2})$ | $L'_{\delta r} = \frac{QSbC'_{l\delta r}}{I_{xx}}$ $(s^{-2})$ | |

Table 7.7 Lateral stability derivatives

**Remarks:**

i) For expressions for $C_{y\beta}$, $C_{n\beta}$, and $C'_{l\beta}$ see Eqs. (7.147), (7.149) and (7.151) respectively.

ii) For expressions for $C_{yp}$ , $C_{np}$ and $C'_{lp}$ . See Eqs.(7.152),(7.155) and (7.157).

iii) For expressions for $C_{yr}$ , $C_{nr}$ , $C'_{lr}$ . See Eqs.(7.162),(7.167) and (7.168a).

**Example 7.1**

A general aviation airplane has the following characteristics.

W = 12232.6 N, $C_D = 0.035 + 0.091\ C_L^2$, $C_{L\alpha} = 4.44\ rad^{-1}$, $\left(\bar{x}_{cg}/\bar{c}\right) = 0.295$

Wing : $S = 17.09\ m^2$ , $b = 10.18\ m$, $C_{L\alpha W} = 4.17\ rad^{-1}$, $\left(x_{ac}/\bar{c} = 0.25\right)$,

$\bar{c} = 1.74\ m$,

Fuselage : $(C_{m\alpha})_{fuselage} = 0.212\ rad^{-1}$

Power : $(C_{m\alpha})_{power} = 0.195\ rad^{-1}$

H.tail : $S_t = 4.73\ m^2$, $C_{L\alpha t} = 3.43\ rad^{-1}$, $\eta_t = 0.9$,

$\eta_t = 0.9$, $l_t = 4.63\ m$, $\frac{d\varepsilon}{d\alpha} = 0.438$

Obtain the values of $C_{X\alpha}$, $C_{Z\alpha}$, $C_{m\alpha}$, $C_{m\dot{\alpha}}$ and $C_{mq}$ in a flight at a velocity of 53.64 m/s at sea level.

**Solution :**

Before calculating the desired quantities, the items like $C_L$, $C_D$, $C_{m\alpha}$ and $C_{D\alpha}$ are evaluated below.

Flight is at sea level, hence, $\rho = 1.225\ kg/m^3$

$$C_L = \frac{2W}{\rho S V^2} = \frac{2\times 12232.6}{1.225\times 17.09\times 53.64^2} = 0.406$$

$$C_D = 0.039 + 0.091\times 0.406^2 = 0.050$$

From Eq.(2.65)

$$C_{m\alpha} = C_{L\alpha w}\left[\frac{x_{cg}}{\bar{c}} - \frac{x_{ac}}{\bar{c}}\right] + (C_{m\alpha})_{f,p} - \bar{V}\eta C_{L\alpha t}\left(1 - \frac{d\varepsilon}{d\alpha}\right)$$

$$= 4.17[0.295 – 0.25] + 0.212 + 0.195 - \frac{4.73}{17.09}\times\frac{4.63}{1.74}\times 0.9\times 3.43(1-0.438)$$

$$= 0.1877 + 0.407 – 1.2777 = - 0.683\ rad^{-1}$$

$$C_{D\alpha} = \frac{2C_L C_{L\alpha}}{\pi A e} = 2\times 0.406\times 4.44\times 0.091 = 0.328$$

The desired quantities are evaluated below.

From Eq.(7.117)

$$C_{X\alpha} = C_L - C_{D\alpha} = 0.406 - 0.328 = 0.078$$

From Eq.(7.121)

$$C_{Z\alpha} = -(C_{L\alpha} + C_D) = -(4.44 + 0.05) = -4.49$$

From Eq.(7.142a)

$$C_{m\dot{\alpha}} = -2C_{L\alpha t}\eta V_H \frac{l_t}{\bar{c}}\frac{d\varepsilon}{d\alpha}$$

$$= 2\times 3.43\times 0.9\times\frac{4.73}{17.09}\times\frac{4.63}{1.74}\times\frac{4.63}{1.74}\times 0.438 = -5.3$$

From Eq.(7.133a)

$$C_{mq} = -2C_{L\alpha t}\eta V_H \frac{l_t}{\bar{c}} = C_{m\dot{\alpha}} / \left(\frac{d\varepsilon}{d\alpha}\right)$$

$$= - 5.3 / 0.438 = -12.1$$

**Answers:**

$C_{X\alpha} = 0.078; C_{Z\alpha} = 4.49; C_{m\dot{\alpha}} = -5.3; C_{mq} = -12.1.$