Chapter 7

Dynamic stability analysis – I – Equations of motion and estimation of stability derivatives - 5

Lecture 26

Topics

7.12.2 $\partial Z / \partial w$

From Fig.7.5:

$Z = -(L\cos\Delta\alpha + D\sin\Delta\alpha)$

Further, $\frac{\partial Z}{\partial w} = \frac{1}{u_0}\frac{\partial Z}{\partial \alpha}$

Hence,

$$\frac{\partial Z}{\partial w} = -\frac{1}{u_0}\{-L\sin\Delta\alpha + \frac{\partial L}{\partial \alpha}\cos\Delta\alpha + D\cos\Delta\alpha + \frac{\partial D}{\partial \alpha}\sin\Delta\alpha\}$$

Taking cos Δα = 1 and ignoring the terms involving sin Δα gives:

$$\frac{\partial Z}{\partial w} = -\frac{1}{u_0}\{\frac{\partial L}{\partial \alpha} + D\} \quad (7.118)$$

$$Z_w = \frac{1}{m}\frac{\partial Z}{\partial w} = -\frac{1}{mu_0}(\frac{\partial L}{\partial \alpha} + D) \quad (7.119)$$

$$= -\frac{1}{mu_0}\{\frac{1}{2}\rho u_0^2\, S C_{L\alpha} + \frac{1}{2}\rho u_0^2\, S C_D\}$$

$$= -\frac{\rho u_0^2 S}{2mu_0}(C_{L\alpha} + C_D) = -\frac{QS}{mu_o}(C_{L\alpha} + C_D) \tag{7.120}$$

Let, $C_{z\alpha} = \frac{Z_w mu_0}{QS}$, then $C_{Z\alpha} = -(C_{L\alpha} + C_D)$ (7.121)

7.12.3 ∂M/∂w

Note that the moment in steady flight (M_0) is zero.

Further, $\frac{\partial M}{\partial w} = \frac{1}{u_0}\frac{\partial M}{\partial \alpha} = \frac{1}{2}\rho S \bar{c} u_0 C_{m\alpha}$ (7.122)

$$M_w = \frac{1}{I_{yy}}\frac{\partial M}{\partial w} = \frac{QS\bar{c}}{u_0 I_{yy}} C_{m\alpha} \tag{7.123}$$

It may be recalled from chapter 2, Eq.(2.65), that:

$$C_{m\alpha} = C_{L\alpha}\left(\frac{x_{cg}}{\bar{c}} - \frac{x_{ac}}{\bar{c}}\right) + (C_{m\alpha})_{f,n,p} - \eta V_H C_{L\alpha t}\left(1 - \frac{d\varepsilon}{d\alpha}\right) \tag{7.124}$$

Remark:

It will be shown in section 8.12 that through this term the static stability influences the dynamic stability.

7.13. Derivatives due to change of ′q′

These derivatives include ∂Z / ∂q and ∂M / ∂q; ∂X / ∂q is negligible. It may be recalled from discussion in section 4.2 that when an airplane has a rate of pitch(q), the components of airplane experience a vertical velocity of magnitude q x l ; 'l' being the longitudinal distance of the component from c.g.. This vertical velocity causes change in angle of attack which gives rise to ΔZ and ΔM and the derivatives ∂Z/∂q and ∂M/∂q. The contribution to the derivative due to 'q' is mainly due to the horizontal tail.

7.13.1 ∂Z / ∂q

Referring to Fig.7.6 the change in lift of horizontal tail due to rotation is given by:

$$\Delta L_t = C_{L\alpha t}\, \Delta\alpha_t\, Q_t\, S_t\, ;\; Q_t = \frac{1}{2}\rho V_t^2 = \eta \frac{1}{2}\rho u_0^2 \tag{7.125}$$

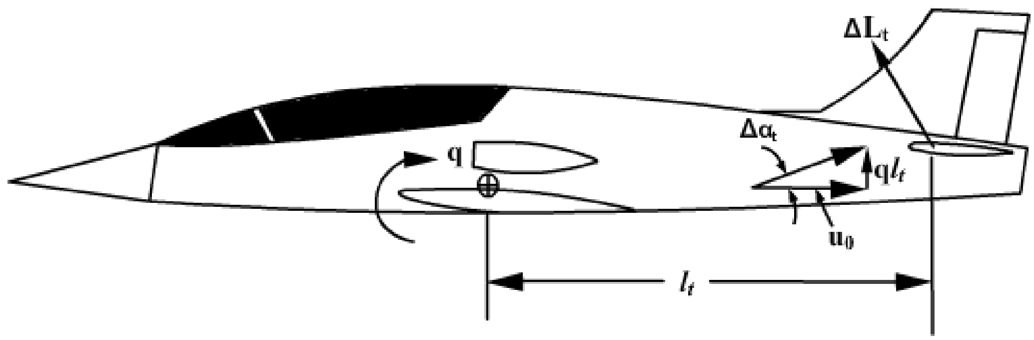

Fig.7.6 Stability derivative due to rate of pitch

Now,

$$\Delta Z = -\Delta L_t = -C_{L\alpha t}\frac{q\,l_t}{u_0}Q_t S_t \tag{7.126}$$

Therefore, $$\frac{\partial Z}{\partial q} = -C_{L\alpha t}\frac{l_t}{u_0}\eta Q S_t \tag{7.127}$$

Let, $$C_Z = \frac{Z}{\frac{1}{2}\rho u_0^2 S}$$

Then, $$\Delta C_Z = \frac{\Delta Z}{\frac{1}{2}\rho u_0^2 S} = \frac{-C_{L\alpha t}\frac{q l_t}{u_0}\eta\frac{1}{2}\rho u_0^2 S_t}{\frac{1}{2}\rho u_0^2 S} = -C_{L\alpha t}\frac{q l_t}{u_0}\eta\frac{S_t}{S}$$

Let, $$C_{Zq} = \frac{\partial C_Z}{\partial(q\bar{c}/2u_0)} = \frac{2u_0}{q\bar{c}}\frac{\partial C_Z}{\partial q}$$

Then, $$C_{Zq} = -\frac{2u_0}{q\bar{c}}C_{L\alpha t}\frac{q\,l_t}{u_0}\eta\frac{S_t}{S} = -2C_{L\alpha t}\eta V_H;\quad V_H = \frac{l_t}{\bar{c}}\frac{S_t}{S} \tag{7.128}$$

$$Z_q = \frac{1}{m}\frac{\partial Z}{\partial q} = -C_{L\alpha t}\,\eta\,\frac{Q}{u_0}\frac{l_t}{\bar{c}}\frac{S_t}{S}\frac{S\bar{c}}{m} = -C_{L\alpha t}\,\eta\,V_H\frac{QS\bar{c}}{mu_0} \tag{7.129}$$

$$Z_q = -C_{Zq}\frac{QS\bar{c}}{2mu_0} \tag{7.130}$$

7.13.2 ∂M/∂q

Refering to Fig.7.6 the change in the moment due to tail on account of the rotation is:

$$(\Delta M_{cg})_t = -l_t \Delta L_t = -l_t C_{L\alpha t} \frac{q\, l_t}{u_0} \eta Q S_t \tag{7.131}$$

Hence, $$\frac{\partial (M_{cg})_t}{\partial q} = -l_t C_{L\alpha t} \frac{l_t}{u_0} \eta Q S_t \tag{7.131a}$$

$$(\Delta C_{mcg})_t = \frac{\Delta M_{cgt}}{\frac{1}{2}\rho u_0^2 S \bar{c}} = \frac{-l_t C_{L\alpha t} \frac{q\, l_t}{u_0} \eta Q S_t}{\frac{1}{2}\rho u_0^2 S \bar{c}} = -V_H \eta C_{L\alpha t} \frac{q\, l_t}{u_0} \tag{7.132}$$

Let, $$(C_{mq})_t = \frac{\partial C_m}{\partial (q\bar{c}/2u_0)} = \frac{2u_0}{\bar{c}} \frac{\partial C_m}{\partial q}$$

Then, $$(C_{mq})_t = -\frac{2u_0}{\bar{c}} C_{L\alpha t}\, \eta V_H \frac{l_t}{u_0} = -2C_{L\alpha t}\, \eta V_H \frac{l_t}{\bar{c}} \tag{7.133}$$

The contributions of wing and body to C_{mq} are small unless the fuselage is long and the wing has low aspect ratio. Hence, for low speed airplanes.

$$\left(C_{mq}\right)_{airplane} \approx \left(C_{mq}\right)_t \tag{7.133a}$$

Combining Eqs. (7.131a) and (7.133a) yields:

$$M_q = \frac{1}{I_{YY}} \frac{\partial M}{\partial q} = C_{mq} \frac{\bar{c}}{2u_0} \frac{Q S \bar{c}}{I_{YY}} \tag{7.134}$$

7.14 Derivatives due to time rate of change of angle of attack ($Z_{\dot{w}}$ and $M_{\dot{w}}$)

According to reference 1.5, chapter 8, when the results of flight test data on stability were compared with the calculated values, some discrepancy was observed. Analysis showed that the terms involving $\dot{\alpha}$ should be included in the equations for Δw and Δq. The reason for this dependence appears due to the following factors.

The flow from the wing takes some time before it reaches the horizontal tail. Hence the downwash angle (ε) at the tail is not the ε that corresponds to α at time t, but that at time (t-Δt). The time lag (Δt) is roughly equal to the time taken by the flow to reach horizontal tail i.e. $\Delta t = l_t / u_0$. The lag in the angle of attack at tail is given by:

$$\Delta\alpha_t = \frac{d\varepsilon}{dt}\Delta t = \frac{d\varepsilon}{dt}\frac{l_t}{u_0} = \frac{d\varepsilon}{d\alpha}\frac{d\alpha}{dt}\frac{l_t}{u_0} = \frac{d\varepsilon}{d\alpha}\frac{\dot{\alpha} l_t}{u_0} \tag{7.135}$$

This lag in angle of attack would cause a change in lift of tail (ΔL_t) given by:

$$\Delta L_t = C_{L\alpha t}\ \Delta\alpha_t\ Q_t\ S_t$$

This change in L_t would mainly change vertical force, Z, and moment M. The change in drag is negligible. The changes in Z and M bring about stability derivaties $Z_{\dot{w}}$ and $M_{\dot{w}}$ which are evaluated below.

$$\Delta C_Z = -\frac{\Delta L_t}{QS} = -C_{L\alpha t}\ \Delta\alpha_t\ \eta\ \frac{S_t}{S} \tag{7.136}$$

$$= -C_{L\alpha t}\frac{d\varepsilon}{d\alpha}\frac{\dot{\alpha}\, l_t}{u_0}\ \eta\ \frac{S_t}{S} \tag{7.137}$$

$$\text{Let, } C_{Z\dot{\alpha}} = \frac{\partial C_Z}{\partial(\dot{\alpha}\bar{c}/2u_0)} = \frac{2u_0}{\bar{c}}\frac{\partial C_Z}{\partial\dot{\alpha}} \tag{7.138}$$

$$\text{Then, } C_{Z\dot{\alpha}} = -\frac{2u_0}{\bar{c}} C_{L\alpha t}\frac{d\varepsilon}{d\alpha}\frac{l_t}{u_0}\ \eta\frac{S_t}{S} = -2C_{L\alpha t}\ \eta\ V_H\frac{d\varepsilon}{d\alpha} \tag{7.139}$$

$$Z_{\dot{w}} = \frac{1}{m}\frac{\partial Z}{\partial\dot{w}} = -C_{L\alpha t}\eta\frac{Q}{u_0}\frac{l_t}{\bar{c}}\frac{S_t}{S}\frac{S\bar{c}}{m} = -C_{Z\dot{\alpha}}\frac{\bar{c}}{2u_0}\frac{QS}{mu_0} \tag{7.140}$$

By a similar argument the change in the moment due to tail owing to the lag is:

$$\Delta M_{cg} = -l_t\ \Delta L_t = -l_t\ C_{L\alpha t}\ \Delta\alpha_t\ Q_t\ S_t \tag{7.141}$$

$$\Delta C_{mcg} = -C_{L\alpha t}\ \eta\ V_H\frac{d\varepsilon}{d\alpha}\frac{\dot{\alpha}\, l_t}{u_0}$$

$$\text{Let, } C_{m\dot{\alpha}} = \frac{\partial C_m}{\partial(\dot{\alpha}\bar{c}/2u_0)} = \frac{2u_0}{\bar{c}}\frac{\partial C_m}{\partial\dot{\alpha}} \tag{7.142}$$

$$C_{m\dot{\alpha}} = -2C_{L\alpha t}\ \eta\ V_H\frac{l_t}{\bar{c}}\frac{d\varepsilon}{d\alpha} = (C_{mq})_{tail}\frac{\partial\varepsilon}{\partial\alpha} \tag{7.142a}$$

$$M_{\dot{w}} = \frac{1}{I_{yy}}\frac{\partial M}{\partial\dot{w}} = -C_{L\alpha t}\ \eta\ \frac{Q}{u_0}\frac{l_t}{\bar{c}}\frac{S_t}{S}\frac{d\varepsilon}{d\alpha}\frac{S\bar{c}}{I_{yy}} = C_{m\dot{\alpha}}\frac{\bar{c}}{2u_0}\frac{QS\bar{c}}{u_0 I_{yy}} \tag{7.143}$$

7.15 Derivatives due to control deflections ($Z_{\delta e}$ and $M_{\delta e}$)

It may be pointed out that the deflection of elevator would change lift on tail. This would result in a force in Z-direction and a moment about c.g. Change in drag is negligible.

$$\Delta Z = -\frac{1}{2}\rho V_t^2 S_t \frac{dC_{Lt}}{d\delta_e}\delta_e = \eta Q S_t \frac{dC_{Lt}}{d\delta_e}\delta_e$$

Or, $\frac{\partial Z_{\delta e}}{\partial \delta_e} = -\eta Q S_t \frac{dC_{Lt}}{d\delta_e}$

Similarly, $\Delta M_{\delta e} = -\eta Q S_t l_t \frac{dC_{Lt}}{d\delta_e}\delta_e$ and $\frac{\partial M}{\partial \delta_e} = -\eta Q S_t l_t \frac{dC_{Lt}}{d\delta_e}$

Let, $C_{Z\delta e} = C_{L\delta e} = -\eta \frac{S_t}{S}\frac{dC_{L_t}}{d\delta_e} = -\eta \frac{S_t}{S} C_{L\alpha t}\tau$

$$\text{Then, } Z_{\delta e} = \frac{1}{m}\frac{\partial Z}{\partial \delta_e} = -C_{Z\delta e}\frac{QS}{m} \tag{7.144}$$

Further, $C_{m\delta e} = -\eta V_H C_{L\alpha t}\tau = -\eta V_H \frac{dC_{L_t}}{d\delta_e}$;

$$\text{Then, } M_{\delta e} = \frac{1}{I_{yy}}\frac{\partial M}{\partial \delta_e} = -C_{m\delta e}\frac{QS\bar{c}}{I_{yy}} \tag{7.145}$$

7.16 Summary of longitudinal stability derivatives

A summary of the longitudinal stability derivatives is given in Table 7.5. The remarks given below in the table, refer to the expressions for the non-dimensional quantities involved in expressions for these derivatives. The list of symbols is given in Table 7.6 for ready reference.

$X_u = \frac{-(C_{Du} + 2C_D)QS}{mu_0}$ (s^{-1})	$Z_u = \frac{-(C_{Lu} + 2C_L)QS}{mu_0}$ (s^{-1})
$M_u = C_{mu} \frac{(QS\bar{c})}{u_0 I_{yy}} \left(\frac{1}{m.s}\right)$	$X_w = \frac{-(C_{D\alpha} - C_L)QS}{mu_0}$ (s^{-1})
$Z_w = \frac{-(C_{L\alpha} + C_D)QS}{mu_0}$ (s^{-1})	$Z_\alpha = u_0 Z_w$ (m/s^2)
$M_w = C_{m\alpha} \frac{(QS\bar{c})}{u_0 I_{yy}} \left(\frac{1}{m.s}\right)$	$M_\alpha = u_0 M_w$ (s^{-2})
$Z_{\dot{w}} = -C_{z\dot{\alpha}} \frac{\bar{c}}{2u_0} QS/(u_0 m)$	$Z_{\dot{\alpha}} = u_0 Z_{\dot{w}}$ (m/s^2)
$M_{\dot{w}} = C_{m\dot{\alpha}} \frac{\bar{c}}{2u_0} \frac{QS\bar{c}}{u_0 I_{yy}}$ (m^{-1})	$M_{\dot{\alpha}} = u_0 M_{\dot{w}}$ (s^{-1})
$Z_q = -C_{Zq} \frac{\bar{c}}{2u_0} \frac{QS}{m}$ (m/s)	$M_q = C_{mq} \left(\frac{\bar{c}}{2u_0}\right)(QS\bar{c}/I_{yy})$ (s^{-1})
$Z_{\delta e} = -C_{Z\delta e} QS/m$ (m/s^2)	$M_{\delta e} = C_{m\delta e} \left(\frac{QS\bar{C}}{I_{yy}}\right)$ (s^{-2})

Table 7.5 Longitudinal stability derivatives

Remarks:

i) C_{Du} , C_{Lu} and C_{mu} are zero at subcritical Mach numbers. See section 7.11.1, 7.11.2 and 7.11.3 for expressions for these quantities at higher Mach numbers.

ii) $C_{D\alpha}$, $C_{L\alpha}$ and $C_{m\alpha}$ are almost always non- zero. See section 7.12.1,7.12.2 and 7.12.3 for expressions for these quantities.

iii) For expressions for C_{zq} and C_{mq} see section 7.13.1 and 7.13.2.

iv) See section 7.14 for expressions for $C_{z\acute{\alpha}}$ and $C_{m\acute{\alpha}}$.

v) See section 7.15 for expressions for $C_{z\delta e}$ and $C_{m\delta e}$.

A	Aspect ratio
b	Wing span
C_D	Drag coefficient in undisturbed flight
C_L	Lift coefficient in undisturbed flight
$C_{L\alpha}$	Airplane lift curve slope
$C_{L\alpha w}$	Wing lift curve slope
$C_{L\alpha t}$	Horizontal tail lift curve slope
$C_{L\alpha v}$	Vertical tail lift curve slope
$\bar{c}$	Mean aerodynamic chord
e	Oswald's span efficiency factor
l_t	Distance from c.g. to tail aerodynamic centre
l_v	Distance from c.g. to vertical tail aerodynamic centre
V_H	Horizontal tail volume ratio
V_v	Vertical tail volume ratio
M_1	Flight Mach number
S	Wing area
S_t	Horizontal tail area
S_v	Vertical tail area
Z_v	Distance from centre of pressure of vertical tail to fuselage centerline
Γ	Wing dihedral angle
Λ	Wing sweep angle
$d\varepsilon/d\alpha$	Change in downwash due to a change in angle of attack
η	Efficiency factor of the horizontal tail
η_v	Efficiency factor of vertical tail
λ	Wing Taper ratio
$d\sigma/ d\beta$	Change in sidewash angle with a change in β

Table 7.6 List of symbols used in expressions for longitudinal and lateral stability derivatives.