Chapter 7

Dynamic stability analysis – I – Equations of motion and estimation of stability derivatives - 4

Lecture 25

Topics

7.8 Expressions for changes in aerodynamic and propulsive forces and moments

The aerodynamic forces and moments and the propulsive force vary with Δu, Δv, Δw, Δp, Δq, Δr, $\Delta\delta_a$, $\Delta\delta_e$ and $\Delta\delta_r$ and their derivatives. According to Ref.1.1, chapter 3, Bryan, who gave the basic frame work of stability analysis in 1911, assumed that these forces and moments can be expressed as functions of the perturbation variables. This can be expressed in the form of a Taylor series as:

ΔX (Δu, Δv, Δw, Δp, Δq, Δr, $\Delta\dot{u}$, $\Delta\dot{v}$, ..$\Delta\delta_a$, $\Delta\delta_e$, $\Delta\delta_r$, $\Delta\delta_T$);

$$= \frac{\partial X}{\partial u}\Delta u + \frac{\partial X}{\partial v}\Delta v + \ldots + \frac{\partial X}{\partial \dot{u}}\Delta\dot{u} + \ldots + \frac{\partial X}{\partial \delta_a}\Delta\delta_a + \frac{\partial X}{\partial \delta_e}\Delta\delta_e + \frac{\partial X}{\partial \delta_r}\Delta\delta_r + \frac{\partial X}{\partial \delta_T}\Delta\delta_T$$

$$+ \text{ higher order terms} \qquad (7.69)$$

Note : $\Delta\delta_T$ is a parameter indicating engine setting.

7.8.1 Simplified expressions for changes in aerodynamic and propulsive forces and moments

The small perturbation equations have been linearized by ignoring the terms containing the powers of perturbation quantities. Continuing the simplification, the higher order terms in Eq.(7.69) are ignored. Further, to avoid unnecessary complications, ΔX, ΔY, ΔZ, $\Delta L'$, ΔM and ΔN are expressed in terms of only a few quantities which directly affect them. Table 7.4 presents the quantities and the perturbation variables on which they depend i.e.

$$\Delta X = \frac{\partial X}{\partial u}\Delta u + \frac{\partial X}{\partial w}\Delta w + \frac{\partial X}{\partial \delta_e}\Delta\delta_e + \frac{\partial X}{\partial \delta_T}\Delta\delta_T \qquad (7.70)$$

$$\Delta Z = \frac{\partial Z}{\partial u}\Delta u + \frac{\partial Z}{\partial w}\Delta w + \frac{\partial Z}{\partial \dot{w}}\Delta\dot{w} + \frac{\partial Z}{\partial q}\Delta q + \frac{\partial Z}{\partial \delta_e}\Delta\delta_e + \frac{\partial Z}{\partial \delta_T}\Delta\delta_T\,;\; q = d\theta/dt \qquad (7.71)$$

$$\Delta M = \frac{\partial M}{\partial u}\Delta u + \frac{\partial M}{\partial w}\Delta w + \frac{\partial M}{\partial \dot{w}}\Delta\dot{w} + \frac{\partial M}{\partial q}\Delta q + \frac{\partial M}{\partial \delta_e}\Delta\delta_e + \frac{\partial M}{\partial \delta_T}\Delta\delta_T \qquad (7.72)$$

$$\Delta Y = \frac{\partial Y}{\partial v}\Delta v + \frac{\partial Y}{\partial p}\Delta p + \frac{\partial Y}{\partial r}\Delta r + \frac{\partial Y}{\partial \delta_r}\Delta\delta_r \qquad (7.73)$$

$$\Delta L' = \frac{\partial L'}{\partial v}\Delta v + \frac{\partial L'}{\partial p}\Delta p + \frac{\partial L'}{\partial r}\Delta r + \frac{\partial L'}{\partial \delta_r}\Delta\delta_r + \frac{\partial L'}{\partial \delta_a}\Delta\delta_a \qquad (7.74)$$

$$\Delta N = \frac{\partial N}{\partial v}\Delta v + \frac{\partial N}{\partial p}\Delta p + \frac{\partial N}{\partial r}\Delta r + \frac{\partial N}{\partial \delta_r}\Delta\delta_r + \frac{\partial N}{\partial \delta_a}\Delta\delta_a \qquad (7.75)$$

Quantity	Dependence on
ΔX	Δu, Δw, $\Delta\delta_e$, $\Delta\delta_T$
ΔZ	Δu, Δw, $\Delta\dot{w}$, Δq, $\Delta\delta_e$, $\Delta\delta_T$
ΔM	Δu, Δw, $\Delta\dot{w}$, Δq, $\Delta\delta_e$, $\Delta\delta_T$
ΔY	Δv, Δp, Δr, $\Delta\delta_r$
$\Delta L'$	Δu, Δp, Δr, $\Delta\delta_r$, $\Delta\delta_a$
ΔN	Δu, Δp, Δr , $\Delta\delta_r$, $\Delta\delta_a$

Table 7.4 Changes in aerodynamic forces and moments and their dependence

Remarks:

i) The simplification of expressing ΔX, ΔZ… ΔN, in terms of only a limited number of variables is possible because of the following reasonable assumptions.

(a) ΔX, ΔW and ΔM are affected only by the variables of longitudinal motion i.e. Δu, Δw, $\Delta\dot{w}$, Δq, $\Delta\delta_e$, $\Delta\delta_T$; the dependence of ΔX on $\Delta\dot{w}$ is ignored (Eq.7.70).

(b) ΔY, $\Delta L'$ and ΔN are dependent only on the variables affecting lateral and directional motions viz. Δv, Δp, Δr and control deflections $\Delta\delta_e$, $\Delta\delta_a$.

ii) These assumptions are valid for conventional airplanes with (a) plane of symmetry, (b) high aspect ratio wings (A>5) and (c) flying at moderate angles of attack. Consult Ref.1.12 chapter 4 for treatment of airplanes with low aspect ratio wings and those operating at high angles of attack.

7.8.2 Stability derivatives

The quantities $\partial X / \partial u$, $\partial X / \partial w$, … $\partial N/ \partial\delta r$, $\partial N/ \partial\delta_a$ in Eqs.(7.70) to (7.75) are called stability derivatives.

7.9 Final form of small perturbation equations

Substituting for ΔX from Eq.(7.70) in Eq.(7.63) yields:

$$m\frac{d\Delta u}{dt}=\frac{\partial X}{\partial u}\Delta u+\frac{\partial X}{\partial w}\Delta w+\frac{\partial X}{\partial \delta_e}\Delta\delta_e+\frac{\partial X}{\partial \delta_T}\Delta\delta_T - mg\ \Delta\theta\ \cos\theta_0 \qquad (7.76)$$

$$\text{Or } \left(m\frac{d}{dt}-\frac{\partial X}{\partial u}\right)\Delta u - \frac{\partial X}{\partial w}\Delta w + mg\cos\theta_0\ \Delta\theta = \frac{\partial X}{\partial \delta_e}\Delta\delta_e + \frac{\partial X}{\partial \delta_T}\Delta\delta_T \qquad (7.77)$$

The following notations are commonly used to simplify the small perturbation equations.

$$X_u = \frac{1}{m}\frac{\partial X}{\partial u}, X_w = \frac{1}{m}\frac{\partial X}{\partial w}, X_{\delta e} = \frac{1}{m}\frac{\partial X}{\partial \delta_e}, X_{\delta r} = \frac{1}{m}\frac{\partial X}{\partial \delta_T}; \qquad (7.78)$$

Using these, Eq.(7.77) can be rewritten as:

$$(\frac{d}{dt} - X_u)\Delta u - X_w \Delta w + g\cos\theta_0 \Delta\theta = X_{\delta e}\Delta\delta_e + X_{\delta T}\Delta\delta_T \qquad (7.79)$$

In a similar manner, the following notations are used to simplify the expressions for ΔZ, ΔM, ΔY, ΔL' and ΔN.

$$Z_u = \frac{1}{m}\frac{\partial Z}{\partial u}, Z_w = \frac{1}{m}\frac{\partial Z}{\partial w}, Z_{\dot{w}} = \frac{1}{m}\frac{\partial Z}{\partial \dot{w}}, Z_{\delta e} = \frac{1}{m}\frac{\partial Z}{\partial \delta_e}, Z_{\delta T} = \frac{1}{m}\frac{\partial Z}{\partial \delta_T}, \qquad (7.80)$$

$$M_u = \frac{1}{I_{yy}}\frac{\partial M}{\partial u}, M_w = \frac{1}{I_{yy}}\frac{\partial M}{\partial w}, \ldots, M_{\delta e} = \frac{1}{I_{yy}}\frac{\partial M}{\partial \delta_e}, M_{\delta T} = \frac{1}{I_{yy}}\frac{\partial M}{\partial \delta_T} \qquad (7.81)$$

$$Y_v = \frac{1}{m}\frac{\partial Y}{\partial v}, Y_r = \frac{1}{m}\frac{\partial Y}{\partial r}, Y_{\delta r} = \frac{1}{m}\frac{\partial Y}{\partial \delta_r} \qquad (7.82)$$

$$L'_v = \frac{1}{I_{xx}}\frac{\partial L'}{\partial v}, L'_p = \frac{1}{I_{xx}}\frac{\partial L'}{\partial p}, L'_r = \frac{1}{I_{xx}}\frac{\partial L'}{\partial r}, L'_{\delta a} = \frac{1}{I_{xx}}\frac{\partial L'}{\partial \delta_a}, L'_{\delta r} = \frac{1}{I_{xx}}\frac{\partial L'}{\partial \delta_r} \qquad (7.83)$$

$$N_v = \frac{1}{I_{zz}}\frac{\partial N}{\partial v}, N_p = \frac{1}{I_{zz}}\frac{\partial N}{\partial p}, N_r = \frac{1}{I_{zz}}\frac{\partial N}{\partial r}, N_{\delta a} = \frac{1}{I_{zz}}\frac{\partial N}{\partial \delta_a}, N_{\delta} = \frac{1}{I_{zz}}\frac{\partial N}{\partial \delta_r} \qquad (7.84)$$

7.9.1 Small perturbation equations for longitudinal motion

The small perturbation equations for longitudinal motion are:

$$(\frac{d}{dt} - X_u)\Delta u - X_w\Delta w + g\cos\theta_0 \Delta\theta = X_{\delta e}\Delta\delta_e + X_{\delta T}\Delta\delta_T \qquad (7.85)$$

$$-Z_u\Delta u + [(1-Z_{\dot{w}})\frac{d}{dt} - Z_w]\Delta w - [(u_0+Z_q)\frac{d}{dt} - g\sin\theta_0]\Delta\theta = Z_{\delta e}\Delta\delta_e + Z_{\delta T}\Delta\delta_T \qquad (7.86)$$

$$-M_u\Delta u - (M_{\dot{w}}\frac{d}{dt} + M_w)\Delta w + (\frac{d^2}{dt^2} - M_q\frac{d}{dt})\Delta\theta = M_{\delta e}\Delta\delta_e + M_{\delta T}\Delta\delta_T \qquad (7.87)$$

7.9.2 Small perturbation equations for lateral motion

The small perturbation equation for lateral motion are:

$$(\frac{d}{dt} - Y_v)\,\Delta v - (u_0 - Y_r)\,\Delta r - g\cos\theta_0\ \ \Delta\varphi = Y_{\delta r}\,\Delta\delta_r \qquad (7.88)$$

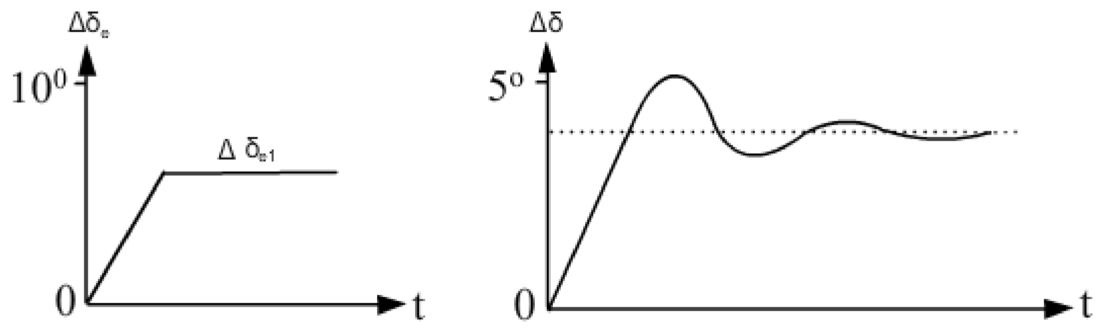
$$-L'_v\Delta v + (\frac{d}{dt} - L'_p)\,\Delta p - [\frac{I_{xz}}{I_{xx}}\frac{d}{dt} + L'_r]\,\Delta r = L'_{\delta a}\,\Delta\delta_a + L'_{\delta r}\,\Delta\delta_r \qquad (7.89)$$

$$-N_v\,\Delta v - (\frac{I_{xz}}{I_{zz}}\frac{d}{dt} + N_p)\,\Delta p - [\frac{d}{dt} - N_r]\Delta r = N_{\delta a}\Delta\delta_a + N_{\delta r}\,\Delta\delta_r \qquad (7.90)$$

7.9.3 Remarks on solutions of small perturbation equations to obtain response of airplane to a disturbance and to the control input

The Eqs.(7.85) to (7.90) constitute the linearized small perturbation equations for longitudinal and lateral motions. Their solution would yield answers to two types of problems namely (a) response to a disturbance and (b) response to a control input. In the case of response to a disturbance, with the control fixed, it is assumed that $\Delta\delta_e$, $\Delta\delta_r$, $\Delta\delta_a$, and $\Delta\delta_T$ are zero. To study the effect of disturbance, it is assumed that one of the parameters from among $\Delta\hat{u}$, $\Delta\hat{w}$,.. .., $\Delta\hat{r}$ is given a small value at time t = 0. Then, the equations are solved with this initial condition. The changes, with time, in the values of the chosen parameters would give information about the dynamic stability. However, it will be pointed out in section 8.2 that it is not necessary to solve the above differential equations to know whether the airplane is stable or not. There is a simpler approach to infer about the stability of the airplane.

In the case of response to the control input, it is assumed that the control deflection is given as a function of time and the solution of the above equations is obtained. For example, it may be prescribed that the elevator deflection $\Delta\delta_e$ changes from zero to a value $\Delta\delta_{e1}$, in a small interval of time and then remains constant (Fig 7.4a). The solution of the equations would give the information about change in angle of attack and the time taken to achieve the final value. Figure 7.4b shows a response wherein the airplane attains the final angle of attack after over-shooting it (final value). In some cases, the final value may be achieved monotonically. However, calculation of response is an involved task. Some indication about this is given in chapter 10.



(a) Elevator deflection (b) Possible response of change in angle of attack.

Fig.7.4 Response to elevator deflection

7.10 Estimation of stability derivatives

The solution of small perturbation equations for longitudinal motion would be taken up in chapter 8 and for the lateral motion in chapter 9. However, to solve these equations the stability derivatives are required. The following subsections deal with their estimation.

7.11 Derivatives due to change of 'Δu'

These derivatives include ∂X / ∂u, ∂Z / ∂u and ∂M / ∂u.

7.11.1 ∂X / ∂u

The changes in ΔX are caused by changes in the drag and the thrust i.e.

$$\Delta X = -\Delta D + \Delta T \tag{7.91}$$

Continuing with the linearized treatment of the problem, the variation of ΔX with Δu is expressed as:

$$\Delta X = -\frac{\partial D}{\partial u}\Delta u + \frac{\partial T}{\partial u}\Delta u \tag{7.92}$$

Or $$\frac{\partial X}{\partial u} = -\frac{\partial D}{\partial u} + \frac{\partial T}{\partial u} = -\frac{\partial}{\partial u}\left(\frac{1}{2}\rho\, u_0^2\, S\, C_D\right) + \frac{\partial T}{\partial u} = -\frac{1}{2}\rho S\left(u_0^2\frac{\partial C_D}{\partial u} + 2u_0 C_D\right) + \frac{\partial T}{\partial u} \tag{7.93}$$

Recall that $X_u = (1 / m)(\partial X / \partial u)$ and let $C_{Du} = \partial C_D / \partial (u / u_0)$

As regards the term, $\partial T/\partial u$ the following may be noted.

a) For gliding flight, T= 0 and hence, $\partial T / \partial u = 0$.

b) For a jet airplane, T is almost constant over small intervals of u and hence, $\partial T/\partial u = 0$.

c) For a piston engined airplane with variable pitch propeller, the THP is nearly constant over a small range of u. Hence,

$T = THP/u$ and consequently $\partial T/\partial u = - THP/u^2 = - D/u$

As regards C_{Du} the following facts may be noted.

a) For subsonic flights with Mach number less than the critical Mach number, the drag coefficient remains constant with Mach number and hence $C_{Du} = 0$.

b) When C_D is a function of Mach number (M_1), C_{Du} is written as:

$C_{Du} = \partial C_D / \partial(u / u_0) = u_0 \partial C_D / \partial(a_0 M_1) = (u_0 / a_0) \partial C_D / \partial M_1 = M_1 \partial C_D / \partial M_1$;

where, a_0 = speed of sound under conditions of undisturbed flight. The symbol M_1 is used for Mach number to avoid confusion with pitching moment (M).

With the above considerations, X_u can be written as:

$$X_u = \frac{1}{m}\frac{\partial X}{\partial u} = -\frac{\rho u_0 S}{2m}(C_{Du} + 2C_D) + \frac{1}{m}\frac{\partial T}{\partial u} \tag{7.94}$$

$$X_u = -\frac{\rho u_0^2 S}{2mu_0}(C_{Du} + 2C_D) + 0 \text{ or } \left(\frac{-D}{mu_0}\right) \tag{7.95}$$

$$= -\frac{QS}{mu_0}(C_{Du} + 2C_D) + 0 \text{ or } \left(\frac{-C_D QS}{mu_0}\right);\ Q = \frac{1}{2}\rho u_0^2 \tag{7.96}$$

$$= -\frac{QS}{mu_0}\{(C_{Du} + 2C_D) + 0 \text{ or } (-C_D)\} \tag{7.97}$$

Following Ref.1.1, chapter 3, two new quantities C_X and C_{Xu} are introduced :

$$C_X = \frac{X}{QS};\quad C_{Xu} = \frac{\partial C_X}{\partial(u/u_0)} = \frac{1}{(1/2)\rho S u_0}\frac{\partial X}{\partial u} \tag{7.98}$$

Consequently,

$$X_u = \frac{1}{m}\frac{\partial X}{\partial u} = -\frac{\rho S u_0^2}{2mu_0}C_{Xu} \tag{7.99}$$

$$C_{Xu} = -\{(C_{Du} + 2C_D) + 0 \text{ or } (-C_D)\} \tag{7.100}$$

7.11.2 ∂Z/∂u

The force in Z-direction is due to the weight and the lift.

$$Z = W - \frac{1}{2}\rho u_0^2 S C_L \quad (7.101)$$

Ignoring the change in weight during the disturbance, gives ($\partial Z/\partial u$) as:

$$\frac{\partial Z}{\partial u} = -\frac{1}{2}\rho S u_0(C_{Lu}+2C_L);\ C_{Lu} = \frac{\partial C_L}{\partial (u/u_0)} \quad (7.102)$$

$$\text{Let, } C_{Zu} = \frac{1}{(1/2)\rho S u_0}\frac{\partial Z}{\partial u};\ \text{Then, } C_{zu} = -(C_{Lu}+2C_L) \quad (7.103)$$

$$\text{Or } Z_u = \frac{1}{m}\frac{\partial Z}{\partial u} = -\frac{\rho S u_0^2}{2mu_0}(C_{Lu}+2C_L) \quad (7.104)$$

$$= -\frac{QS}{mu_0}(C_{Lu}+2C_L) \quad (7.105)$$

It may be added that:

a) At low subsonic Mach number C_{Lu} can be neglected.

b) At sub-critical Mach numbers Ref 1.1, chapter 3 states that dC_L / dM_1 can be calculated using Prandtl -Glauert rule applicable to airfoils. However, Ref.1.12, chapter 4 suggests that:

$$C_{Lu} = M_1\frac{dC_L}{dM_1} = M_1\,\alpha\,\frac{dC_{L\alpha}}{dM_1} \quad (7.106)$$

The term $C_{L\alpha}$ as a function of Mach number is given as:

$$C_{L\alpha} = \frac{2\pi A}{2+\sqrt{\frac{A^2\beta_1^2}{K^2}\frac{(1+\tan^2\Lambda_{c/2})}{\beta_1^2}+4}}\ \text{in rad}^{-1} \quad (7.107)$$

where, A = Aspect ratio of wing, $\Lambda_{c/2}$ = sweep of the mid-chord line, $\beta_1 = (1-M_1^2)^{1/2}$ and K = (lift curve slope of airfoil) / 2π.

Remark:

The stability derivative C_{Lu} for Boeing-747 is calculated at $M_1 = 0.8$ in Appendix ‘C’. For this purpose $C_{L\alpha}$ is evaluated at $M_1 = 0.82$ and $M_1 = 0.78$. Using these two values $dC_{L\alpha} / dM$ is calculated at $M_1 = 0.8$.

7.11.3 ∂M/∂u

Noting that $M = \frac{1}{2}\rho u_0^2\, S\, \bar{c}\, C_{mcg}$ gives: (7.108)

$$\frac{\partial M}{\partial u} = \frac{\partial}{\partial u}\left(\frac{1}{2}\rho S u_0^2\, \bar{c}\, C_{mcg}\right)$$

$$= \frac{1}{2}\rho S\, \bar{c}\, \left(u_0^2 \frac{\partial C_{mcg}}{\partial u} + 2\, u_0 C_{mcg}\right)$$

But, $C_{mcg} = 0$ in equilibrium flight. Consequently,

$$\frac{\partial M}{\partial u} = \frac{1}{2}\rho\, S\, \bar{c}\, u_0 C_{mu} \quad \text{where, } C_{mu} = \frac{\partial C_M}{\partial (u/u_0)} \qquad (7.109)$$

Analogous to Eq.(7.105), C_{mu} can be expressed as:

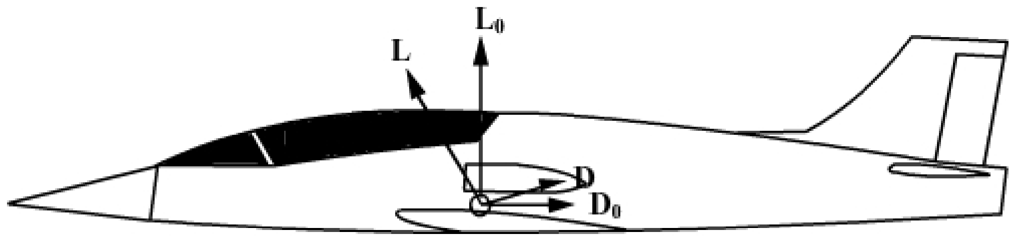
$$C_{mu} = M_1 \frac{\partial C_m}{\partial M_1} = M_1\, \alpha \frac{dC_{m\alpha}}{dM_1} \quad \text{where, } M_1 \text{ is Mach number} \qquad (7.110)$$

$$\text{Hence,}\quad M_u = \frac{1}{I_{yy}}\frac{\partial M}{\partial u} = \frac{Q\, S\, \bar{c}}{u_0\, I_{yy}} C_{mu} \qquad (7.111)$$

Changes in $C_{m\alpha}$ with M_1 are found out by obtaining $C_{m\alpha}$ at nearby Mach numbers (see Appendix ‘C’). The value is also affected by elastic bending of the wing and fuselage.

7.12 Derivatives due to change of ′Δw′

These include ∂X / ∂w, ∂Z / ∂w and ∂M / ∂w. The discussion in this subsection is based on Ref.7.2, chapter 4. It may be noted that in the undisturbed flight, X_s- axis is along the flight direction. i.e. $\mathbf{V} = u_0\, \mathbf{i}$, $w_0 = 0$, $v_0 = 0$. After the disturbance, the airplane has Δw (Fig.7.5). Thus, the relative wind makes an angle $\Delta\alpha = \Delta w / u_0$. The lift (L) and drag (D) are now perpendicular and parallel respectively to the relative velocity ($\mathbf{V}_R$) as shown in Fig.7.5.



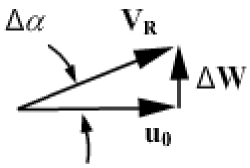


Fig.7.5 Stability derivatives due changes of w

7.12.1 ∂X / ∂w

From Fig.7.5 ,

$$X = L\sin\Delta\alpha - D\cos\Delta\alpha \tag{7.112}$$

Further, $$\frac{\partial X}{\partial w} = \frac{1}{u_0}\frac{\partial X}{\partial \alpha} \tag{7.113}$$

Hence, $$\frac{\partial X}{\partial w} = \frac{1}{u_0}\{L\cos\Delta\alpha + \frac{\partial L}{\partial \alpha}\sin\Delta\alpha + D\sin\Delta\alpha - \frac{\partial D}{\partial \alpha}\cos\Delta\alpha\} \tag{7.114}$$

Since, $\Delta\alpha$ is small, $\cos\Delta\alpha = 1$ and terms involving $\sin\Delta\alpha$ are ignored.

Hence, $$X_w = \frac{1}{m}\frac{\partial X}{\partial w} = \frac{1}{mu_0}(L - \frac{\partial D}{\partial \alpha}) \tag{7.115}$$

$$= \frac{1}{mu_0}\{\frac{1}{2}\rho u_0^2 SC_L - \frac{\partial}{\partial \alpha}(\frac{1}{2}\rho u_0^2 SC_D)\}$$

$$= \frac{\rho u_0^2 S}{2mu_0}(C_L - C_{D\alpha}) = -\frac{QS}{mu_0}(C_{D\alpha} - C_L) \tag{7.116}$$

Let, $C_{X\alpha} = \frac{X_w m u_0}{QS}$, Then, $C_{X\alpha} = C_L - C_{D\alpha}$ (7.117)

Note: For Mach number less than the critical Mach number the drag polar is given by:

$$C_D = C_{D0} + \frac{C_L^2}{\pi Ae}.\ \text{Hence,}\ C_{D\alpha} = \frac{2C_L}{\pi Ae}C_{L\alpha}.$$