Chapter 7

Dynamic stability analysis – I – Equations of motion and estimation of stability derivatives - 3

Lecture 24

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7.6 Equation of motion in stability axes system

As mentioned earlier the scalar form of the equations of motion is obtained by resolving all the forces and moments along a single axes system. Stability axes system is used for this purpose. It is assumed that in the equilibrium state the airplane is in an unaccelerated climb at angle θ_0 to the horizontal. It may be pointed out that the unaccelerated level flight is a special case with θ_0= 0. When stability axes system is used, the axis OX_s is in the plane of symmetry and along the direction of flight. The following may be noted.

(a) When the airplane encounters a disturbance it (airplane) is displaced but the orientation of the stability axes remains fixed in space.

(b) If u,v,w are the components of the velocity vector along the stability axes these components in the undisturbed flight , u_0,v_0 and w_0 are respectively u_0= $|\mathbf{V}|$, v_0 = 0 and w_0 = 0.

(c) The moments of inertia of the airplane may be calculated about some convenient axes system, generally the reference body axes system. When the stability axes system is used, the moments of inertia would have to be obtained about these axes (see, for example, Ref.1.10, appendix ‘B.12’ for necessary formulae).

The relationships between various axes systems are already derived in the previous section. Let X, Y and Z be the components of the aerodynamic and propulsive forces along the OX_s, OY_s and OZ_s axes respectively.

The gravitational force is mg $\mathbf{k_h}$. Noting from Eq. (7.43a) that

$\mathbf{k_h} = -\sin\theta\, \mathbf{i_s} + \cos\theta \sin\varphi\, \mathbf{j_s} + \cos\theta\cos\varphi\, \mathbf{k_s}$, the components of the gravitational force along OX_s, OY_s and OZ_s axes are

$$
\begin{aligned}
(F_x)_{gravity} &= -\, mg \sin\theta \\
(F_y)_{gravity} &= mg \cos\theta \sin\varphi \\
(F_z)_{gravity} &= mg \cos\theta \cos\varphi
\end{aligned}
\tag{7.47}
$$

Let L′, M and N be the moments due to the aerodynamic and propulsive forces about the OX_s, OY_s and OZ_s axes. Using Eqs.(7.19), (7.34) to (7.36) and (7.47) the following equations of motion are obtained in the scalar form. They are actually the components of forces and moments along OX_s, OY_s and OZ_s axes. It may be noted that for an airplane with plane of symmetry, two of the products of inertia namely D and F are zero.

$$m(\dot{u} + qw - rv) = X - mg\sin\theta \tag{7.48}$$

$$m(\dot{v} + ru - pw) = Y + mg\cos\theta \sin\varphi \tag{7.49}$$

$$m(\dot{w} + pv - qu) = Z + mg\cos\theta\cos\varphi \tag{7.50}$$

$$I_{xx}\,\dot{p} - I_{xz}\,\dot{r} + (I_{zz} - I_{yy})\,qr - I_{xz}\,pq = L' \tag{7.51}$$

$$I_{yy}\dot{q} + (I_{xx} - I_{zz})\,rp + I_{xz}(p^2 - r^2) = M \tag{7.52}$$

$$I_{zz}\dot{r} - I_{xz}\dot{p} + (I_{yy} - I_{xx})\,pq - I_{xz}qr = N \tag{7.53}$$

Remarks:

i) As noted earlier u, v and w are the components of velocity along the stability axes systems. In the undisturbed flight these components are denoted by u_0,v_0 and w_0. Since, stability axes system is used, $v_0 = w_0 = 0$.

ii) Equations (7.48) to (7.53) are the six equations for the six unknown quantities viz. u, v, w, p, q and r. It may be recalled that a rigid airplane is a system with six degrees of freedom. These six equations form the set of governing equations for the dynamics of a rigid airplane.

7.7 Equations governing motion after a small disturbance

At outset the following points may be noted.

1)The quantities u, v, w, p, q and r are the six dependent variables for the motion of a rigid airplane.

2)In the undisturbed (or the equilibrium) state these quantities have the values u_0,v_0,w_0, p_0,q_0 and r_0.

3)When the airplane is disturbed, these quantities would become $u_0 + \Delta u$, $v_0 + \Delta v$, $w_0 + \Delta w$, $p_0 + \Delta p$, $q_0 + \Delta q$ and $r_0 + \Delta r$.

4)To know whether the airplane is stable, the variations of Δu, Δv, Δw, Δp, Δq and Δr with time have to be obtained.

5)One of the approaches to obtain the equations for Δu, Δv,…,Δr is the small perturbation theory. Elements of this theory and the derivation of the equations are presented in the following parts of this section and sections 7.8 and 7.9.

7.7.1 Introduction to small perturbation theory

The small perturbation theory is used commonly to analyse the stability of various phenomena. The basic ideas are as follows.

I) Each system is described by (a) a certain number of variables called the state variables and (b) by a set of governing equations. For example, in the case of the motion of a rigid airplane, (a) the state variables are u, v, w, p, q, and r and (b) the motion is governed by six equations namely Eqs.(7.48) to (7.53). The state variables have certain values namely u_0,v_0,w_0, p_0,q_0 and r_0 in the undisturbed state. After the disturbance, these values will change to $u_0 + \Delta u$, $v_0 + \Delta v$, $w_0 + \Delta w$, $p_0 + \Delta p$, $q_0 + \Delta q$ and $r_0 + \Delta r$. The deviations Δu, Δv, …. Δr are functions of time.

II) It is assumed that the deviations Δu, Δv, …. Δr, caused due to a disturbance, are small compared to the reference values in the undisturbed state. In this connection, notes 2,3 and 4, after Eq.(7.54) below, clarify the matter further.

The quantities Δu, Δv, …. Δr are called perturbations. For the sake of generality a variable may be denoted by 'f '. It has a value f_0 in the undisturbed state. After the disturbance, it becomes $f_o+\Delta f$. The quantity Δf is called the perturbation in f about f_0. The deviation "Δf" is a function of time.

III)The equations for the perturbations are obtained by substituting $(u_0 + \Delta u)$, $(v_0 + \Delta v)$, $(w_0 + \Delta w)$, $(p_0 + \Delta p)$, $(q_0 + \Delta q)$ and $(r_0 + \Delta r)$ in place of u, v, w, p, q, and r respectively in the governing equations. It is evident that these equations will involve (a) u_0,v_0,w_0, p_0,q_0 and r_0 and (b) Δu, Δv, Δw, Δp, Δq and Δr ; of these quantities u_0,v_0,w_0, p_0,q_0 and r_0 are known.

The resulting equations are, in general, non-linear as they may involve products of the perturbations or their powers.These equations are simplified by assuming that the perturbations are small and consequently, the products of the perturbations or their second and higher powers are ignored. This way the equations for Δu, Δv, Δw, Δp, Δq and Δr become linear. This is made clearer when the motion of the airplane is dealt with in the next subsection.

IV) The equations for Δu, Δv, Δw, Δp, Δq and Δr are solved and examined for the evolution of these quantities with time. If all of them ultimately are zero, then the system is stable. If anyone of them does not do so, then the system is unstable. Figure 1.6 shows six possible ways in which Δu etc. may change with time.

7.7.2 Application of small perturbation theory to the motion of airplane

Regarding the problem of the stability of an airplane, the application of small perturbation theory is done with the help of the following steps.

I) It is prescribed that u, v, w, p, q, r, X,Y,Z, L′, M, N and the control deflections δ_a , δ_e and δ_r have the values u_0,v_0,w_0,p_0,q_0,r_0,X_0,Z_0, L'_0, M_0,N_0 , δ_{a0} , δ_{e0} and δ_{r0} respectively in the equilibrium state; some of these values could be zero. After the disturbance these quantities are expressed as:

$u = u_0 + \Delta u$; $v = v_0 + \Delta v$; $w = w_0 + \Delta w$;

$p = p_0 + \Delta p$; $q = q_0 + \Delta q$; $r = r_0 + \Delta r$;

$X = X_0 + \Delta X$; $Y = Y_0 + \Delta Y$; $Z = Z_0 + \Delta Z$;   (7.54)

$L' = L'_0 + \Delta L'$; $M = M_0 + \Delta M$; $N = N_0 + \Delta N$;

$\delta_a = \delta_{a0} + \Delta\delta_a$; $\delta_e = \delta_{e0} + \Delta\delta_e$; $\delta_r = \delta_{r0} + \Delta\delta_r$;

$\theta = \theta_0 + \Delta\theta$; θ_0 is the angle of climb in undisturbed flight.

Note:1) An airplane in steady climb is also in an equilibrium flight. The angle of climb, here denoted by θ, was denoted by γ in performance analysis.

2) The quantities Δu, Δv, and Δw are much smaller as compared to u_0 i.e. Δu, Δv, Δw << u_0.

3) The quantities (l Δp), (l Δq), (l Δr) << u_0, where l is a reference length; Δp, Δq, Δr have been multiplied by 'l' so that the product has dimensions of velocity.

4) ΔX, ΔY, ΔZ are small as compared to the lift developed by the airplane.

5) The reference flight is steady and is in the plane of symmetry. In this case :

$v_0 = p_0 = q_0 = r_0 = \varphi_0 = \psi_0 = 0$

where, φ_0 = bank angle and ψ_0 = angle of yaw.

The products of inertia D and F are also zero in this case.

6) Further, the x-axis is the same as x_s- axis, and the velocity component 'w_0' is zero.

II) To obtain the equations for Δu, Δv, Δw, Δp, Δq and Δr, the instantaneous quantities (u_0 + Δu) , (v_0+ Δv), ……., ($\delta_{r0} + \Delta\delta_r$) given by Eq.(7.54) are substituted in the equations of motion viz Eqs.(7.48) to (7.53).

Consider Eq.(7.48). With the introduction of instantaneous quantities, this equation is changed to :

$$m\left[\frac{d(u_0+\Delta u)}{dt} + (q_0+\Delta q)(w_0+\Delta w) - (r_0+\Delta r)(v_0+\Delta v)\right]$$

$$= X_0 + \Delta X - mg\sin(\theta_0 + \Delta\theta) \tag{7.55}$$

Note: $w_0 = v_0 = q_0 = r_0 = 0$.

Five more equations are obtained by substituting instantaneous quantities in Eqs.(7.49) to (7.53) . These five equations plus Eq.(7.55) are the six equations for Δu, Δv, Δw, Δp, Δq and Δr. These are non-linear equations as they involve products of the unknown like (Δq x Δw), (Δr x Δv) etc.

7.7.3 Linearization of equations for perturbation quantities

As mentioned in the previous paragraph the equations for perturbation quantities are non-linear. They can be linearised by ignoring, the products involving unknown quantities, as small. However, the linearization process can be appreciated better by writing the equations in non-dimensional form. The following non-dimensionalization scheme is used.

Let, $$\Delta\hat{u} = \frac{\Delta u}{u_0};\ \Delta\hat{v} = \frac{\Delta v}{u_0};\Delta\hat{w} = \frac{\Delta w}{u_0};\Delta\hat{q} = \frac{l\Delta q}{u_0};\Delta\hat{r} = \frac{l\Delta r}{u_0};\ \hat{t} = \frac{t u_0}{l} \qquad (7.56)$$

Remark:

It may be pointed out that for the purpose of non-dimensionalization, the reference quantities are: (a) u_0, the flight velocity in undisturbed flight, as the reference velocity, (b) l, as the reference length; it can be l_t or b as the case may be. (c) the reference time is (l/u_o). Following observations are added about the choice of reference time.

The reference quantities l and u_0 arise as different airplanes would have different lengths and would fly at different velocities. However, there is no imposed time scale in the stability problem. Hence, the particular variations of Δu etc. with time, obtained for an airplane, are the result of the length and velocity scales for the airplane. Consequently, the reference time (t_{ref}) is obtained from a combination of l and u_o i.e. $t_{ref} = l / u_0$. The non-dimensional time ($\hat{t}$) in Eq.(7.56) is given as:

$$\hat{t} = t / t_{ref} = t\, u_0 / l.$$

Replacing Δu by $\Delta\hat{u} \times u_0$, Δq by $\Delta\hat{q}\frac{u_0}{l}$ and so on in Eq.(7.55) , the following form is obtained.

$$m\left[\frac{d(u_0 + \Delta\hat{u}\times u_0)}{d\left(\hat{t}\frac{l}{u_0}\right)} + \left(\Delta\hat{q}\frac{u_0}{l}\right)(\Delta\dot{w}\times u_0) - \left(\Delta\hat{r}\frac{u_0}{l}\right)(\Delta\hat{v}\times u_0)\right]$$
$$= X_0 + \Delta X - mg\sin(\theta_0 + \Delta\theta) \qquad (7.57)$$

Or $\frac{mu_0^2}{l}[\frac{d(1+\Delta\hat{u})}{d\hat{t}} + \Delta\hat{q}\times\Delta\hat{w} - \Delta\hat{r}\times\Delta\hat{v}] = X_0 + \Delta X - mg\sin(\theta_0 + \Delta\theta)$ (7.58)

Dividing Eq.(7.58) by $\frac{mu_0^2}{l}$, the non-dimensional form of Eq. (7.55) is:

$$\frac{d(1+\Delta\hat{u})}{d\hat{t}} + \Delta\hat{q}\times\Delta\hat{w} - \Delta\hat{r}\times\Delta\hat{v} = \frac{lX_0}{mu_0^2} + \frac{l\Delta X}{mu_0^2} - \frac{gl}{u_0^2}\sin(\theta_0 + \Delta\theta) \quad (7.59)$$

In Eq.(7.59) the products ($\Delta\hat{q}\times\Delta\hat{w}$) and ($\Delta\hat{r}\times\Delta\hat{u}$) appear. It may be pointed out that though Δu, Δv, Δw, Δq, Δr etc. may take some arbitrary values, the quantities $\Delta\hat{u}, \Delta\hat{v}$, $\Delta\hat{w}$, $\Delta\hat{q}$, $\Delta\hat{r}$ etc. are, by definition, much smaller than unity or infinitesimal of the first order. Hence, $\Delta\hat{u}$, $\Delta\hat{q}$ etc. are not negligible but the products like ($\Delta\hat{q}\times\Delta\hat{w}$), ($\Delta\hat{r}\times\Delta\hat{v}$) are much smaller than unity i.e. second order infinitesimal or roughly speaking of the order of 0.01. Hence, these products are negligible as compared to 1. It may also be pointed out that for $\Delta\hat{u}$, $\Delta\hat{v}$ etc. are dependent variables and they cannot be ignored when they appear individually. With these simplifications, Eq.(7.59) is reduced to:

$$\frac{d\,\Delta\hat{u}}{d\hat{t}} = \frac{lX_0}{mu_0^2} + \frac{l\Delta X}{mu_0^2} - \frac{gl}{u_0^2}\sin(\theta_0 + \Delta\theta) \quad (7.60)$$

Further,

$$\sin(\theta_0 + \Delta\theta) = \sin\theta_0\cos\Delta\theta + \cos\theta_0\sin\Delta\theta$$

Though θ_0 may not be small, Δθ is small and hence cos Δθ ≈ 1 and sin Δθ = Δθ. Consequently,

$$\sin(\theta_0 + \Delta\theta) \approx \sin\theta_0 + \Delta\theta\cos\theta_0 \quad (7.61)$$

Reverting back to the dimensional form, Eq.(7.60) becomes:

$$m\frac{d\Delta u}{dt} = X_0 + \Delta X - mg(\sin\theta_0 + \Delta\theta\cos\theta_0) \quad (7.62)$$

It may be recalled that the undisturbed flight is a steady climb at θ_0 . From the performance analysis of climb flight, the equation of motion along OX_s axis is:

$$T_0 - D_0 - W\sin\theta_0 = 0 \quad (7.62a)$$

where, T_0 and D_0 are thrust and drag in steady climb.

Noting that:

(i) X_0 = (Aerodynamic + propulsive force) in an undisturbed flight = $T_0 - D_0$

(ii) W = mg

Eq.(7.62a) gives:

$X_0 - mg \sin \theta_0 = 0$.

Consequently, Eq.(7.62) is reduced to:

$$m\frac{d\Delta u}{dt} = \Delta X - mg\ \Delta\theta\ \cos\theta_0$$

$$\text{Or } m\ \Delta\dot{u} = \Delta X - mg\ \Delta\theta\ \cos\theta_0\ ;\ \Delta\dot{u} = \frac{d\Delta u}{dt} \tag{7.63}$$

The steps adopted to derive Eq.(7.63) from Eq.(7.48), can be followed to derive the small perturbation forms of Eqs.(7.50), (7.52),(7.49)(7.51) and (7.53). The derivations are given as an exercise for the reader.

Recalling from subsection 1.3.5, the dynamic stability analysis can be simplified by splitting the six degrees of freedom system into two systems. Each of them with three degrees of freedom namely longitudinal motion and lateral motion. The final form of the equations for longitudinal and lateral motions are given below and presented in Table 7.3. Some clarifications regarding the derivations are given as remarks after the table.

7.7.4 Small perturbation equations for longitudinal motion

$$m\ \Delta\dot{u} = \Delta X - mg\ \Delta\theta\ \cos\theta_0 \tag{7.63}$$

$$m(\Delta\dot{w} - u_0\ \Delta q) = \Delta Z - mg\ \Delta\theta\ \sin\theta_0, \tag{7.64}$$

$$I_{yy}\Delta\dot{q} = \Delta M\ ;\ \Delta\dot{q} = \frac{d(\Delta q)}{dt} \tag{7.65}$$

7.7.5 Small perturbation equations for lateral motion

$$m\ (\dot{v} + u_0\ \Delta r) = \Delta Y + mg \cos\theta_0\ \Delta\varphi \tag{7.66}$$

$$I_{xx}\ \Delta\dot{p} - I_{xz}\ \Delta\dot{r} = \Delta L' \tag{7.67}$$

$$I_{xz}\ \Delta\dot{p} + I_{zz}\ \Delta\dot{r} = \Delta N \tag{7.68}$$

where, $\Delta\dot{p} = \dfrac{d(\Delta p)}{dt}$ and $\Delta\dot{r} = \dfrac{d(\Delta r)}{dt}$

Equations of motion	Small perturbation form
Longitudinal motion	
$m(\dot{u} + qw - rv) = X - mg\sin\theta$	$m\ \Delta\dot{u} = \Delta X - mg\,\Delta\theta\cos\theta_0$
$m(\dot{w} + pv - qu) = Z + mg\cos\theta\cos\varphi$	$m(\Delta\dot{w} - u_0\Delta q) = \Delta Z - mg\,\Delta\theta\sin\theta_0$
$I_{yy}\dot{q} + (I_{xx} - I_{zz})rp + I_{xz}(p^2 - r^2) = M$	$I_{yy}\Delta\dot{q} = \Delta M$
Lateral motion	
$m(\dot{v} + ru - pw) = Y + mg\cos\theta\sin\varphi$	$m(\Delta\dot{v} + u_0\Delta r) = \Delta Y + mg\cos\theta_0\ \Delta\varphi$
$I_{xx}\ \dot{p} - I_{xz}\ \dot{r} + (I_{zz} - I_{yy})\ q\ r - I_{xz}\ pq = L'$	$I_{xx}\ \Delta\dot{p} - I_{xz}\ \Delta\dot{r} = \Delta L'$
$I_{zz}\ \dot{r} - I_{xz}\ \dot{p} + (I_{yy} - I_{xx})p\ q - I_{xz}q\ r = N$	$I_{xz}\ \Delta\dot{p} + I_{zz}\ \Delta\dot{r} = \Delta N$

Table 7.3 Equations of motion and their small perturbation form

Remarks: The steps involved in deriving Eq.(7.63) from Eq.(7.48) are given above. The following aspects are kept in mind while deriving Eqs.(7.64),(7.65),(7.66),(7.67) and (7.68) from Eqs.(7.50), (7.52), (7.49), (7.51) and (7.53) respectively.

i) Equation (7.50):

(a) The term 'pv' would become (Δp x Δv) and would drop out as product of two small quantities.

(b) The term 'qu' would become Δq (u_0 + Δu). This has two terms (Δq x u_0) and (Δq x Δu). The term (Δq x Δu) drops out as product of small quantities. However, (Δq x u_0) is retained as u_0 is not small. In Eq.(7.86) of subsection 7.9.1, $\dfrac{d\Delta\theta}{dt}$ is used in place of Δq and in Eq.(7.87), $\dfrac{d^2\Delta\theta}{dt^2}$ is used in place of $\Delta\dot{q}$.

(c) In the undisturbed flight of steady climb at angle θ_0, the equation of motion is: L_0 - W cos θ_0= 0, L_0 = lift in undisturbed flight which in the present notation becomes Z_0 + mg cos θ_0 = 0. Taking these aspects into account and noting that

φ is zero in steady climb, the linearized small perturbation form is given as Eq.(7.64).It is compared with Eq.(7.50) in Table 7.3.

ii) Equation (7.52):

(a) The term 'r p' would become (Δr x Δp) and can be neglected as product of two small terms.

(b) The term '$(r^2 - p^2)$' would become $(\Delta r)^2 - (\Delta p)^2$ and can be neglected as powers of small terms.

(c) In steady flight pitching moment is zero and hence M_0 is zero.

The linearized small perturbation form is given as Eq.(7.65) and compared with Eq.(7.52) in Table 7.3.

iii) Equation (7.49):

(a) The term 'ru' would become Δr ($u_0 + \Delta u$) . The term (Δr x Δu) is neglected but the term $u_0\Delta r$ is retained.

(b) The term 'pw' becomes (Δp x Δw) and is neglected.

(c) In steady flight there is no sideslip and hence Y_0 is zero.

The linearized small perturbation form is given as Eq. (7.66) and compared with Eq.(7.49) in Table 7.3.

iv) Equation (7.51):

(a) The term 'qr' becomes (Δq x Δr) and is neglected.

(b) The term 'pq' become (Δp x Δq) and is neglected.

(c) The rolling moment (L') is zero in steady flight.

The linearized small perturbation form is given as Eq.(7.67) and compared with Eq.(7.51) in Table 7.3.

v) Equation (7.53):

The terms involving 'pq' and 'qr' drop out as insignificant and N_0 is zero in steady flight. The linearized small perturbation form is given as Eq.(7.68) and compared with Eq.(7.53) in Table 7.3.