Chapter 7

Dynamic stability analysis – I – Equations of motion and estimation of stability derivatives - 2

Lecture 23

Topics

7.4 Axes systems

Different types of coordinate systems used in flight dynamic analysis, are briefly explained in this section. It is assumed that the airplane has a plane of symmetry.The following axes systems are used in the dynamic stability analysis presented here.

1. Ground axes system ($EX_e Y_e Z_e$)
2. Local horizon system ($OX_h Y_h Z_h$)
3. Body axes system and its special cases namely reference body axes system ($OX_b Y_b Z_b$) wind axes system ($OX_w Y_w Z_w$) and stability axes system ($OX_s Y_s Z_s$)

Some of these axes systems are shown in Fig.7.2a and 7.2b and are defined as follows.

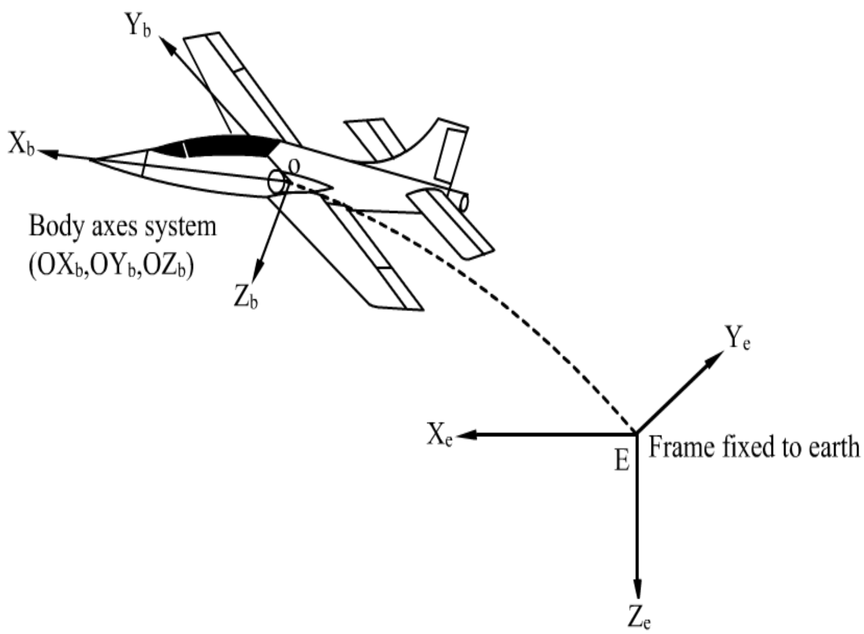

Fig.7.2a Ground axes system ($OX_eY_e Z_e$) and reference body axes system ($OX_bY_bZ_b$).

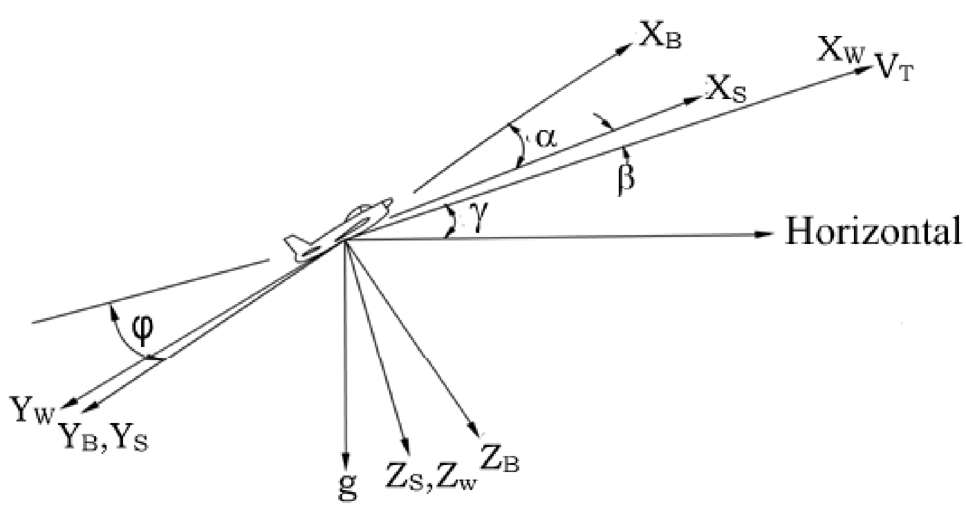

Fig.7.2b Reference Body axes system, wind axes system and stability axes system (Adapted from Ref.2.4)

7.4.1. Ground axes system

This axes system is fixed with respect to earth and defined as follows. The origin E is a point on the earth's surface. The EZ_e- axis is vertical and positive downwards. The EX_e- and EY_e- axes are contained in a horizontal plane and are directed in such a way that $EX_eY_eZ_e$ forms a right handed system (Fig.7.2a).

Remark:

Reference 1.12, chapter 4 calls the aforesaid ground axes system as navigational system. The point E is chosen in such a way that it is beneath the c.g. of the airplane at the start of the flight. The axis OX_e could be towards the local north and the axis OY_e towards the local east. It may be added that there are other definitions of navigational system.

7.4.2. Local horizon system

This system has its origin at a point O on the airplane which lies in the plane of symmetry and generally the c.g. of the airplane. The axes OX_h, OY_h and OZ_h are parallel to EX_e, EY_e and EZ_e respectively.

7.4.3. Body axes system

Any axes system fixed to the airplane and moving with it is called body axes system.

7.4.4. Reference body axes system

Following Ref.7.1, the axes system used for prescribing the moments and products of inertia of the airplane is called "Reference body axes system" and is denoted OX_b Y_b Z_b. In this system the axis OX_b is contained in the plane of symmetry, and is positive forward (towards nose of the airplane). Generally, it coincides with the fuselage reference line (FRL). OZ_b is perpendicular to OX_b and is contained in the plane of symmetry. OZ_b is positive downwards for normal flight attitude of the airplane. OY_b is perpendicular to the plane of symmetry and is directed in such a way that $OX_bY_bZ_b$ is a right handed triad (Fig.7.2a).

7.4.5 Wind axes system

This is also a body axes system in which the axis OX_w is tangent to the flight path in the undisturbed state and positive in forward direction. OZ_w is perpendicular to OX_w and contained in plane of symmetry and positive downwards for normal flight attitude of the airplane. OY_w is perpendicular to both OX_w and OZ_w and is directed in such a way that $OX_wY_wZ_w$ is a right handed system (Fig.7.2b).

7.4.6 Stability axes system

This is also a body axes system and denoted by $OX_sY_sZ_s$. The axis OX_s lies in the plane of symmetry and if the undisturbed flight is with no sideslip ($\beta = 0$), then it (OX_s axis) points in the flight direction. If $\beta \neq 0$ then OX_s coincides with the projection, in the plane of symmetry, of the flight velocity vector. The axis OY_s is perpendicular to the plane of symmetry, positive in starboard(right) direction, and the axis OZ_s points downwards in such a way that $OX_sY_sZ_s$ is a right handed system.

It may be pointed out that (i) the component of velocity along OZ_s axis is zero (ii) the angle between OX_b and OX_s is the angle of attack and (iii) if $\beta = 0$, then drag and lift are in directions opposite to OX_s and OZ_s axes respectively.

Remarks:

i) Figure 7.2 b shows an airplane in a flight along a straight line. The airplane is banked at angle 'φ'. The velocity vector (**V**) makes an angle γ to the horizontal. The airplane has an angle of attack 'α' and sideslip 'β'. The system $OX_bY_bZ_b$ is the reference body axes system with the axis OX_b in the plane of symmetry along fuselage reference line (FRL), the axis OZ_b perpendicular to OX_b in the plane of symmetry and the axis OY_b perpendicular to both OX_b and OZ_b. The system $OX_wY_wZ_w$ is the wind axes system with the axis OX_w in the flight direction, the axis OZ_w perpendicular to OX_w in the plane of symmetry and the axis OY_w perpendicular to both OX_w and OZ_w. The system $OX_sY_sZ_s$ is the stability axes system with the axis OX_s as the projection of **V** in the plane of symmetry, the axis OZ_s perpendicular to OX_s in the plane of symmetry, and OY_s perpendicular to both OX_s and OZ_s.

ii) Generally the sideslip (β) is zero in the undisturbed flight and the systems $OX_wY_wZ_w$ and $OX_sY_sZ_s$ coincide with each other (Fig.7.2c).

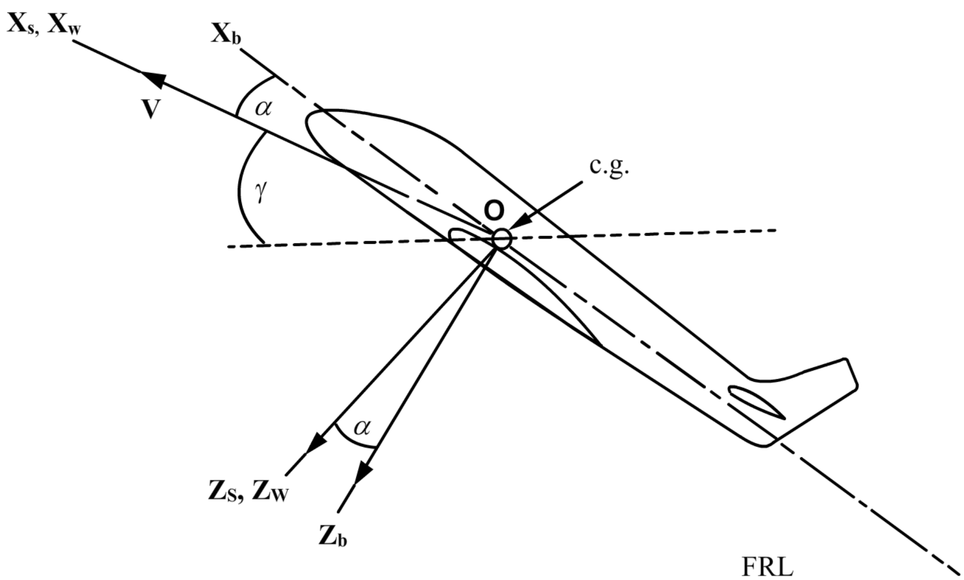

Fig.7.2c Axes systems when β = 0

7.5. Relationships between the various axes systems

As mentioned in section 7.3, the forces **T**, **A** and m**g** are defined in axes system appropriate for each of these quantities. However, to get the scalar form of Eq.(7.37), these forces need to be expressed in a single axes system.

For this purpose, the relationships between various axes systems are needed.These are derived in the next three subsections.

7.5.1 Relation between ground axes and local horizon system

Since, the ground axes system and local horizon systems are parallel to each other, the unit vectors in the two systems are related as: $\mathbf{i_h} = \mathbf{i_e}$, $\mathbf{j_h} = \mathbf{j_e}$, $\mathbf{k_h} = \mathbf{k_e}$. In matrix form the relationship can be expressed as:

$$\begin{pmatrix} \mathbf{i}_h \\ \mathbf{j}_h \\ \mathbf{k}_h \end{pmatrix} = \begin{pmatrix} \mathbf{i}_e \\ \mathbf{j}_e \\ \mathbf{k}_e \end{pmatrix} \tag{7.38}$$

7.5.2 Relation between a general body axes system and the local horizon system

At the outset, it may be recalled that any system of axes fixed to the body and moving with it is a body axes system. Hence, the three axes systems namely, the reference body axes system, the stability axes system and the wind axes system are all special types of body axes systems. In this subsection $OX_{gb}Y_{gb}Z_{gb}$ is taken as a general body axes system which can be any of the aforesaid three special cases. The unit vectors along OX_{gb}, OY_{gb} and OZ_{gb} are $\mathbf{i}_{gb}$, $\mathbf{j}_{gb}$ and $\mathbf{k}_{gb}$ respectively.

In general, the transformation between $\mathbf{i}_{gb}$, $\mathbf{j}_{gb}$ and $\mathbf{k}_{gb}$ and $\mathbf{i}_h$, $\mathbf{j}_h$ and $\mathbf{k}_h$ can be written as:

$$\begin{aligned}\mathbf{i}_{gb} &= a_{11}\mathbf{i}_h + a_{12}\mathbf{j}_h + a_{13}\mathbf{k}_h \\ \mathbf{j}_{gb} &= a_{21}\mathbf{i}_h + a_{22}\mathbf{j}_h + a_{23}\mathbf{k}_h \\ \mathbf{k}_{gb} &= a_{31}\mathbf{i}_h + a_{32}\mathbf{j}_h + a_{33}\mathbf{k}_h\end{aligned} \tag{7.39}$$

The set of Eqs.(7.39) indicates that nine quantities namely, $a_{11}, a_{12}, \ldots, a_{33}$ are needed to prescribe the transformation between the two axes systems. These nine quantities are called direction cosines. However, the following six relationships exist between the unit vectors in the two coordinate systems viz. $\mathbf{i}_{gb}.\mathbf{i}_{gb} = 1$, $\mathbf{i}_{gb}.\mathbf{j}_{gb} = 0$, $\mathbf{i}_{gb}.\mathbf{k}_{jgb} = 0$; $\mathbf{j}_{gb}.\mathbf{j}_{gb} = 1$, $\mathbf{j}_{gb}.\mathbf{k}_{gb} = 0$; $\mathbf{k}_{gb}.\mathbf{k}_{gb} = 1$. Thus, only three quantities are needed to describe the transformation from one coordinate system to another.

There are various ways to arrive at these three quantities. In the approach called Eulerian angles, one coordinate system is rotated through three angles about three different axes to finally arrive at the new system. This approach is generally followed in flight dynamics. The steps are as follows.

a) First, rotate $OX_h Y_h Z_h$ system about OZ_h axis through yaw angle ψ to get a new coordinate system $OX_1Y_1Z_1$ (Fig.7.3). The unit vectors in this system are $\mathbf{i_1}$, $\mathbf{j_1}$ and $\mathbf{k_1}$. The unit vectors in the two axes systems are related as:

$$\mathbf{i_1} = \cos\psi\,\mathbf{i_h} + \sin\psi\,\mathbf{j_h}$$

$$\mathbf{j_1} = -\sin\psi\,\mathbf{i_h} + \cos\psi\,\mathbf{j_h}$$

$$\mathbf{k_1} = \mathbf{k_h}.$$

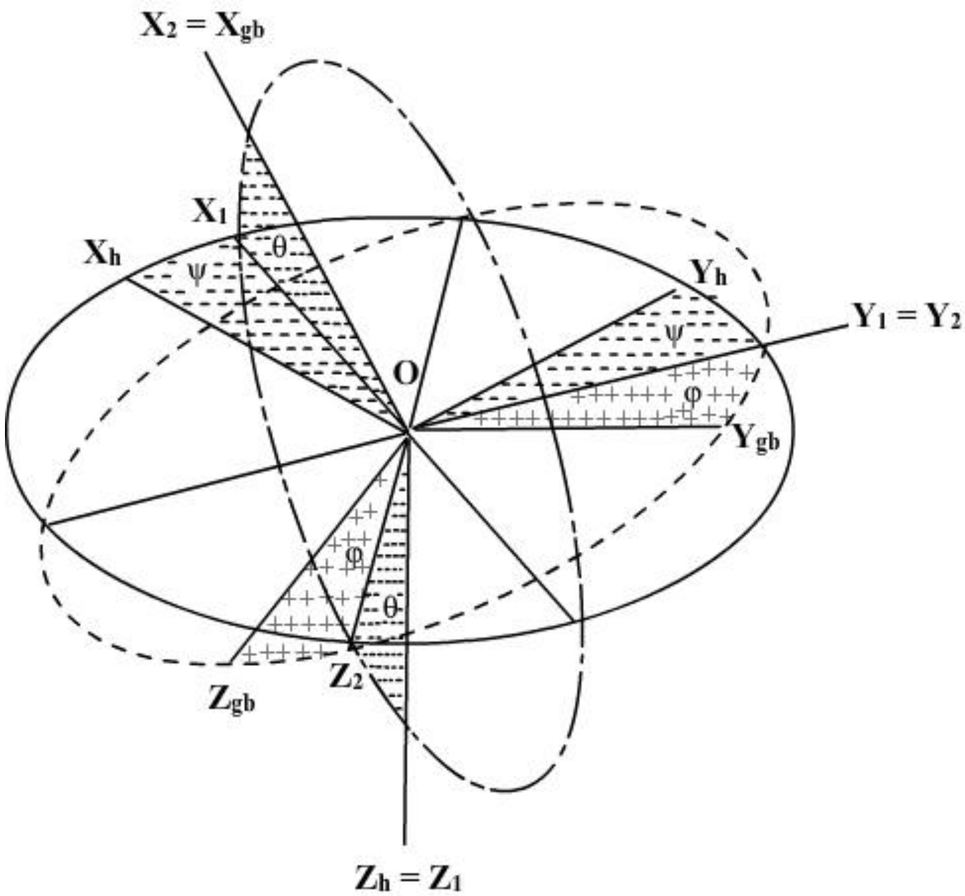

Fig.7.3 Successive rotations of local horizon system to arrive at body axes system

In matrix notation, these relations can be written as:

$$\begin{pmatrix} \mathbf{i}_1 \\ \mathbf{j}_1 \\ \mathbf{k}_1 \end{pmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \mathbf{i}_h \\ \mathbf{j}_h \\ \mathbf{k}_h \end{pmatrix} \qquad (7.40)$$

b) Next, rotate $OX_1Y_1Z_1$ system through pitch angle θ about OY_1 axis to get the coordinate system $OX_2Y_2Z_2$ (Fig.7.3). The unit vectors in this system are are $\mathbf{i_2}$, $\mathbf{j_2}$ and $\mathbf{k_2}$.The unit vectors in the two systems are related as:

$$\begin{pmatrix} \mathbf{i}_2 \\ \mathbf{j}_2 \\ \mathbf{k}_2 \end{pmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{pmatrix} \mathbf{i}_1 \\ \mathbf{j}_1 \\ \mathbf{k}_1 \end{pmatrix} \qquad (7.41)$$

c) Finally, rotate $OX_2Y_2Z_2$ system through bank angle φ about OX_2 axis to arrive at the system $OX_{gb}Y_{gb}Z_{gb}$. The unit vectors in this system are denoted by $\mathbf{i}_{gb}$, $\mathbf{j}_{gb}$ and $\mathbf{k}_{gb}$.The unit vectors in the two axes systems are related as:

$$\begin{pmatrix} \mathbf{i}_{gb} \\ \mathbf{j}_{gb} \\ \mathbf{k}_{gb} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \begin{pmatrix} \mathbf{i}_2 \\ \mathbf{j}_2 \\ \mathbf{k}_2 \end{pmatrix} \qquad (7.42)$$

Combining Eqs.(7.40),(7.41) and (7.42), results in the following relationship:

$$\begin{pmatrix} \mathbf{i}_{gb} \\ \mathbf{j}_{gb} \\ \mathbf{k}_{gb} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \mathbf{i}_h \\ \mathbf{j}_h \\ \mathbf{k}_h \end{pmatrix} \qquad (7.43)$$

After multiplying the matrices in Eq.(7.43), the relationship between the unit vectors $\mathbf{i}_{gb}$, $\mathbf{j}_{gb}$ and $\mathbf{k}_{gb}$ and the unit vectors $\mathbf{i}_h$, $\mathbf{j}_h$ and $\mathbf{k}_h$ is obtained. It is presented in Table 7.1, The relationship between $\mathbf{i}_h$, $\mathbf{j}_h$ and $\mathbf{k}_h$ and $\mathbf{i}_{gb}$, $\mathbf{j}_{gb}$ and $\mathbf{k}_{gb}$ can also be obtained from the same table. For example:

$$\mathbf{k}_h = -\sin\theta\, \mathbf{i}_{gb} + \sin\varphi\, \cos\theta\, \mathbf{j}_{gb} + \cos\varphi\, \cos\theta\, \mathbf{k}_{gb}. \qquad (7.43a)$$

$$\mathbf{k}_{gb} = (\cos\varphi\, \sin\theta \cos\Psi + \sin\varphi\, \sin\Psi)\, \mathbf{i}_h + (\cos\varphi\, \sin\theta \sin\Psi - \sin\varphi\, \cos\Psi)\, \mathbf{j}_h + (\cos\varphi\, \cos\theta)\, \mathbf{k}_h \qquad (7.43b)$$

Remarks:

i) The angles ψ, θ and φ are the Eulerian angles.

ii) See Ref. 1.12 chapter 4 for other ways of transforming axes systems.

	i_{gb}	j_{gb}	k_{gb}
i_h	$\cos\theta \cos\Psi$	$\sin\varphi \sin\theta \cos\Psi - \cos\varphi \sin\Psi$	$\cos\varphi \sin\theta \cos\Psi + \sin\varphi \sin\Psi$
j_h	$\cos\theta \sin\Psi$	$\sin\varphi \sin\theta \sin\Psi + \cos\varphi \cos\Psi$	$\cos\varphi \sin\theta \sin\Psi - \sin\varphi \cos\Psi$
k_h	$-\sin\theta$	$\sin\varphi \cos\theta$	$\mathrm{Cos}\varphi \cos\theta$

Table 7.1 Relation between unit vectors of the general body axes system and and the local horizon systems

7.5.3 Relation between wind axes system ($OX_wY_wZ_w$) and reference body axes systems ($OX_bY_bZ_b$)

This relationship is described here for the sake of completeness. It may be noted that the OX_b and OZ_b axes are contained in the plane of symmetry of the airplane. Hence, only two angular rotations namely, the sideslip angle β and the angle of attack α are needed to determine the relative orientation of the two sets of axes.

I) First, rotate $OX_w Y_w Z_w$ system about OZ_w axis through angle β to get a new coordinate system $OX_3Y_3Z_3$ (Fig.7.3). The unit vectors in this system are $\mathbf{i_3}$, $\mathbf{j_3}$ and $\mathbf{k_3}$. The unit vectors in the two axes systems are related as:

$$\begin{pmatrix} \mathbf{i}_3 \\ \mathbf{j}_3 \\ \mathbf{k}_3 \end{pmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \mathbf{i}_w \\ \mathbf{j}_w \\ \mathbf{k}_w \end{pmatrix} \qquad (7.44)$$

II) Then, rotate the $OX_3 Y_3 Z_3$ system through an angle α about the OY_3 axis to arrive at the body axes system $OX_b Y_b Z_b$. The unit vectors in this system are $\mathbf{i_b}$, $\mathbf{j_b}$ and $\mathbf{k_b}$. The unit vectors in the two axes systems are related as :

$$\begin{pmatrix} \mathbf{i}_b \\ \mathbf{j}_b \\ \mathbf{k}_b \end{pmatrix} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{pmatrix} \mathbf{i}_3 \\ \mathbf{j}_3 \\ \mathbf{k}_3 \end{pmatrix} \qquad (7.45)$$

Combining Eqs.(7.44) and (7.45), results in the following relationship:

$$\begin{pmatrix} \mathbf{i_b} \\ \mathbf{j_b} \\ \mathbf{k_b} \end{pmatrix} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \mathbf{i_w} \\ \mathbf{j_w} \\ \mathbf{k_w} \end{pmatrix} \tag{7.46}$$

After multiplying the matrices in Eq.(7.46), the relationship between the unit vectors $\mathbf{i}_b$, $\mathbf{j}_b$ and $\mathbf{k}_b$ and the unit vectors $\mathbf{i}_w$, $\mathbf{j}_w$ and $\mathbf{k}_w$ is obtained. It is presented in Table 7.2, The relationship between $\mathbf{i_w}$, $\mathbf{j_w}$ and $\mathbf{k_w}$ and $\mathbf{i}_b$, $\mathbf{j}_b$ and $\mathbf{k}_b$ can also be obtained from the same table.

	$\mathbf{i}_b$	$\mathbf{j}_b$	$\mathbf{k}_b$
$\mathbf{i}_w$	cosα cosβ	sinβ	sin φ cosβ
$\mathbf{j}_w$	-cosα sinβ	cosβ	-sinα sinβ
$\mathbf{k}_w$	-sinα	0	cosα

Table 7.2 Relation between the unit vectors of the wind axes and the reference body axes systems

Remark: It may be noted that the angles α and β are the angle of attack and the angle of sideslip.

The relationships between the various axes systems used in flight dynamics are thus obtained. The equations of motion in stability axes system are derived now.