Chapter 7

Dynamic stability analysis – I – Equations of motion and estimation of stability derivatives

(Lectures 22 to 27)

Keywords : Equations of motion for a rigid body in vector and scalar form ; forces and moments acting on an airplane ; various axes systems in flight dynamics and relationships between them ; Euler angles ; equation of motion in stability axes system ; linearised small perturbation equations for longitudinal and lateral motions ; stability derivatives.

Topics

Chapter 7

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Lecture 22

Topics

7.1 Introduction

As mentioned in earlier chapters a system is said to have static stability when, after being given a small disturbance, it has a tendency to return to the equilibrium position. To analyse the static stability, the moments brought about immediately after the disturbance are only to be considered. However, for a system to be dynamically stable it must finally return to the equilibrium position. Thus, to examine the dynamic stability, the motion following a disturbance or an intended control input needs to be analysed. This motion is called response. However, an airplane is a system with six degrees of freedom and obtaining the response is a difficult task. Presently, computer programs are available to analyse response of such systems. However, in this introductory course the equations of motion are derived and simplified forms are obtained. Subsequently, the conditions that ensure dynamic stability are deduced without solving the equations. The topic of response would be touched upon, in chapter 10, with examples of simpler systems.

The treatment of dynamic stability can be divided into the following topics.

1. Derivation of the equations of motion, in vector and scalar forms, for a rigid body.

2. Axes systems used for describing airplane motion.
3. Equations of motion in stability axes system.
4. Equation of motion with small perturbation.
5. Linearization and decomposition of small perturbation equations.
6. Stability derivatives.
7. Solution of the equations of motion for longitudinal motion.
8. Equations of motion in state variable form
9. Stability diagrams
10. Approximations of longitudinal motion
11. Lateral dynamic stability

The topics 1 to 6, from the above list are dealt with in this chapter. Topics 7-10 are discussed in chapter 8 and topic 11 is covered in chapter 9.

7.2 Equations of motion in vector and scalar forms

The equations of motion are obtained by applying the Newton's second law to the motion of airplane. For this purpose the airplane is treated as a rigid body which is translating as well as rotating. This motion is decomposed as:
(a) translation of the c.g. of the airplane with reference to an inertial frame which is taken as a frame fixed at a point on the earth and
(b) rotation with respect to the inertial system of a body axes system, attached to the airplane. The linear velocity, vector and the angular velocity vector are resolved along the body axes system (see Eqs.7.12a and 7.12b).

It may be pointed out that the use of a body axes system has the advantage that the moments of inertia and products of inertia calculated with respect to this system would remain almost constant. They(moments of inertia) may change slightly during the flight due to the consumption of fuel or deflection of control surfaces.

To apply Newton's second law to the motion of an airplane, requires an expression for the acceleration of an elemental mass ′dm′ located at a point on the body. An expression for it (acceleration), in the context of the aforesaid decomposition, is derived in the next subsection.

7.2.1 Acceleration of a particle on a rigid body

Let the fixed frame of reference be $EX_eY_e\,Z_e$ with origin at E. Let the body fixed coordinated system attached to the airplane be OXYZ (Fig.7.1). Let the unit vectors along OX, OY and OZ axes be **i, j** and **k**. It may be mentioned that the quantities in bold letters are vectors. With reference to the co-ordinate systems, the position vector of the elemental mass 'dm' at point P on the body can be expressed as:

$$\mathbf{EP}=\mathbf{EO}+\mathbf{OP}=\mathbf{EO}+x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}; \tag{7.1}$$

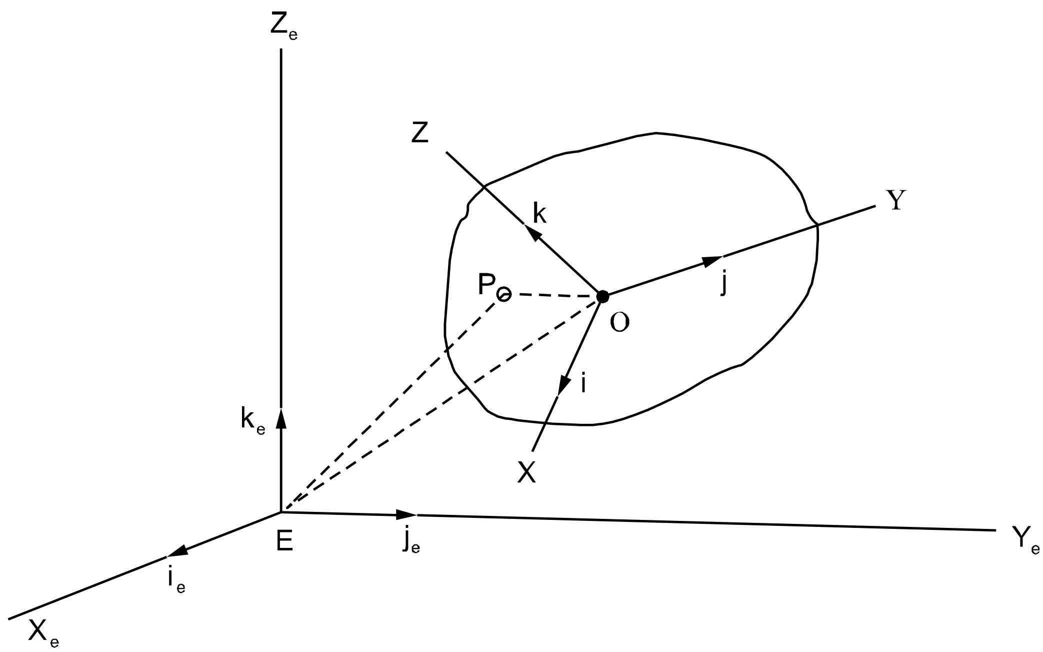

Fig 7.1 Motion of a rigid body

The velocity of an elemental mass 'dm' at point P is the time derivative of its position vector i.e.

$$\mathbf{V}=\frac{d(\mathbf{EP})}{dt}=\frac{d(\mathbf{EO})}{dt}+\frac{d(\mathbf{OP})}{dt}=\frac{d(\mathbf{EO})}{dt}+\frac{d(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})}{dt}$$

$$=\quad \mathbf{V_0}+\frac{dx}{dt}i+\frac{dy}{dt}j+\frac{dz}{dt}k+x\frac{d\mathbf{i}}{dt}+y\frac{d\mathbf{j}}{dt}+z\frac{d\mathbf{k}}{dt} \tag{7.2}$$

$$\mathbf{V}_0 = \frac{\mathrm{d}(\mathbf{EO})}{\mathrm{dt}} \tag{7.3}$$

For a rigid body, the distance between any two points is constant. Hence,

$$\mathrm{dx / dt = dy / dt = dz / dt = 0.} \tag{7.4}$$

When a system of axes rotates with angular velocity **ω**, then

$$\frac{\mathrm{d}\mathbf{i}}{\mathrm{dt}} = \boldsymbol{\omega} \times \mathbf{i}\ ;\ \frac{\mathrm{d}\mathbf{j}}{\mathrm{dt}} = \boldsymbol{\omega} \times \mathbf{j};\frac{\mathrm{d}\mathbf{k}}{\mathrm{dt}} = \boldsymbol{\omega} \times \mathbf{k} \tag{7.5}$$

Hence, $\frac{\mathrm{d}\mathbf{OP}}{\mathrm{dt}} = \boldsymbol{\omega} \times (\mathrm{x}\,\mathbf{i} + \mathrm{y}\,\mathbf{j} + \mathrm{z}\,\mathbf{k}) = \boldsymbol{\omega} \times \mathbf{OP}$

Thus, the velocity of the elemental mass at point P can be written as:

$$\mathbf{V} = \mathbf{V}_0 + \boldsymbol{\omega} \times (\mathbf{x}\ \mathrm{i} + \mathbf{y}\ \mathrm{j} + \mathbf{z}\ \mathrm{k}) \tag{7.6}$$

$$= \mathbf{V}_0 + \boldsymbol{\omega} \times \mathbf{OP} \tag{7.7}$$

The acceleration is the rate of change of velocity i.e.

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{V}}{\mathrm{dt}} = \frac{\mathrm{d}(\mathbf{V}_0 + \boldsymbol{\omega} \times \mathbf{OP})}{\mathrm{dt}} \tag{7.8}$$

$$\mathbf{a} = \mathbf{a}_0 + \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{dt}} \times \mathbf{OP} + \boldsymbol{\omega} \times \frac{\mathrm{d}\mathbf{OP}}{\mathrm{dt}}\ ;\ \mathbf{a}_0 = \frac{\mathrm{d}\mathbf{V}_0}{\mathrm{dt}} \tag{7.9}$$

Replacing $\frac{\mathrm{d}\mathbf{OP}}{\mathrm{dt}}$ by **ω x OP**, the acceleration (**a**) of the elemental mass at point P can be expressed as:

$$\mathbf{a} = \mathbf{a}_0 + \dot{\boldsymbol{\omega}} \times \mathbf{OP} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{OP}) \tag{7.10}$$

7.2.2 Vector form of equations of motion

Applying Newton's second law of motion, the equations of motion is vector form are:

$$\mathbf{F} = \int_m \mathbf{a}\ \mathrm{dm} \tag{7.11}$$

$$\mathbf{M} = \int_m \mathbf{OP} \times \mathbf{a}\ \mathrm{dm} \tag{7.12}$$

7.2.3. Scalar form of equations of motion

Though the vector form of the equations of motion is compact, the equations in scalar form are needed to solve them. This is done by writing the expressions for acceleration, forces and moments as sum of their x-, y-, and z-components and then equating the expressions for the three components on the two sides of Eqs.(7.11) and (7.12). While deriving the scalar form, some terms would not be resolved into components, since they would vanish on integration over the mass of the body.

Let the origin on the body have a velocity ($\mathbf{V_0}$) i.e.

$$\mathbf{V_0} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \tag{7.12a}$$

and angular velocity, $\boldsymbol{\omega} = p\,\mathbf{i} + q\,\mathbf{j} + r\,\mathbf{k}$ (7.12b)

Then, the velocity of any point on the body as shown earlier (Eq.7.7) is:

$$\mathbf{V = V_0 + \boldsymbol{\omega} \times OP}$$

The acceleration of the point P, from Eq.(7.9), is :

$$\mathbf{a} = \frac{d(u\mathbf{i} + v\,\mathbf{j} + w\mathbf{k})}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{OP} + \boldsymbol{\omega} \times \frac{d\mathbf{OP}}{dt} \tag{7.13}$$

Expanding the terms on the right hand side of Eq.(7.13) gives:

$$\begin{aligned}\mathbf{a} &= \dot{u}\,\mathbf{i} + \dot{v}\,\mathbf{j} + \dot{w}\,\mathbf{k} + u(\boldsymbol{\omega} \times \mathbf{i}) + v\,(\boldsymbol{\omega} \times \mathbf{j}) + w\,(\boldsymbol{\omega} \times \mathbf{k}) \\ &\quad + \{\dot{p}\,\mathbf{i} + \dot{q}\,\mathbf{j} + \dot{r}\,\mathbf{k} + p(\boldsymbol{\omega} \times \mathbf{i}) + q\,(\boldsymbol{\omega} \times \mathbf{j}) + r\,(\boldsymbol{\omega} \times \mathbf{k})\} \times \mathbf{OP} \\ &\quad + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{OP})\end{aligned} \tag{7.14}$$

Equation (7.11) can now be written as:

$$\begin{aligned}\mathbf{F} &= \int_m \{\dot{u}\mathbf{i} + \dot{v}\mathbf{j} + \dot{w}\mathbf{k} + u(\boldsymbol{\omega} \times \mathbf{i}) + v(\boldsymbol{\omega} \times \mathbf{j}) + w(\boldsymbol{\omega} \times \mathbf{k})\}dm \\ &+ \{\dot{p}\,\mathbf{i} + \dot{q}\,\mathbf{j} + \dot{r}\,\mathbf{k} + p(\boldsymbol{\omega} \times \mathbf{i}) + q(\boldsymbol{\omega} \times \mathbf{j}) + r(\boldsymbol{\omega} \times \mathbf{k})\} \times \int_m \mathbf{OP}\,dm \\ &+ \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \int_m \mathbf{OP}\,dm)\end{aligned} \tag{7.15}$$

Taking 'O' as the center of mass, gives:

$$\int_m \mathbf{OP}\,dm = 0 \tag{7.16}$$

Hence, Eq.(7.15) simplifies to:

$$\begin{aligned}\mathbf{F} &= m\,[\dot{u}\mathbf{i} + \dot{v}\mathbf{j} + \dot{w}\mathbf{k} + u\,(r\,\mathbf{j} - q\,\mathbf{k}) + v\,(p\mathbf{k} - r\,\mathbf{i}) + w\,(q\mathbf{i} - p\mathbf{j})] \\ &= m\,(\dot{u} + qw\text{-}r\,v)\,\mathbf{i} + m(\dot{v}+r\,u\text{-}pw)\,\mathbf{j} + m(\dot{w}+pv\text{-}qu)\,\mathbf{k}\end{aligned} \tag{7.17}$$

Let **F** be expressed as:

$$\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k} \tag{7.18}$$

Equating Eqs.(7.17) and (7.18) gives the force equations in scalar form as:

$$F_x = m(\dot{u} + q\,w - r\,v)$$
$$F_y = m(\dot{v} + r\,u - p\,w) \qquad (7.19)$$
$$F_z = m(\dot{w} + p\,v - q\,u)$$

The scalar form of moment equations is now derived. The equation in the vector form is:

$$\mathbf{M} = \int_m \mathbf{OP} \times \mathbf{a}\,dm$$

$$\mathbf{M} = \int_m \mathbf{OP} \times [\mathbf{a_0} + \dot{\boldsymbol{\omega}} \times \mathbf{OP} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{OP})]\ dm \qquad (7.20)$$

This equation involves several vector products. Their components are obtained through the following steps.

$$\boldsymbol{\omega} = p\,\mathbf{i} + q\,\mathbf{j} + r\,\mathbf{k}$$

Differentiating this expression with time and using Eq.(7.5) gives :

$$\begin{aligned}\dot{\boldsymbol{\omega}} &= \dot{p}\mathbf{i} + \dot{q}\mathbf{j} + \dot{r}\mathbf{k} + p(\boldsymbol{\omega} \times \mathbf{i}) + q(\boldsymbol{\omega} \times \mathbf{j}) + r(\boldsymbol{\omega} \times \mathbf{k}) \\ &= \dot{p}\mathbf{i} + \dot{q}\mathbf{j} + \dot{r}\mathbf{k} + p(r\,\mathbf{j} - q\mathbf{k}) + q(p\mathbf{k} - r\,\mathbf{i}) + r(q\mathbf{i} - p\mathbf{j}) \\ &= \dot{p}\mathbf{i} + \dot{q}\mathbf{j} + \dot{r}\mathbf{k}\end{aligned} \qquad (7.21)$$

$$\begin{aligned}\boldsymbol{\omega} \times \mathbf{OP} &= (p\mathbf{i} + q\mathbf{j} + r\mathbf{k}) \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\ &= (qz - ry)\mathbf{i} + (rx - pz)\mathbf{j} + (py - qx)\mathbf{k}\end{aligned} \qquad (7.22)$$

Similarly, using Eq.(7.21) the following result is obtained.

$$\dot{\boldsymbol{\omega}} \times \mathbf{OP} = (\dot{q}\,z - \dot{r}\,y)\mathbf{i} + (\dot{r}\,x - \dot{p}\,z)\mathbf{j} + (\dot{p}y - \dot{q}x)\mathbf{k} \qquad (7.23)$$

$$\begin{aligned}\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{OP}) &= (p\mathbf{i} + q\mathbf{j} + r\mathrm{k}) \times \{(qz - ry)\mathrm{i} + (rx - pz)\mathrm{j} + (py - qx)\mathrm{k}\} \\ &= (pqy - q^2x - r^2x + prz)\mathbf{i} + (qrz - r^2y - p^2y + qpx)\mathbf{j} \\ &\quad + (rpx - p^2z - q^2z + rqy)\mathbf{k}\end{aligned} \qquad (7.24)$$

$$\begin{aligned}\mathbf{OP} \times (\dot{\boldsymbol{\omega}} \times \mathbf{OP}) &= (\dot{p}y^2 - \dot{q}xy - \dot{r}xz + \dot{p}z^2)\mathbf{i} + (\dot{q}z^2 - \dot{r}yz - \dot{p}yx + \dot{q}x^2)\mathbf{j} \\ &\quad + (\dot{r}x^2 - \dot{p}zx - \dot{q}zy + \dot{r}y^2)\mathbf{k}\end{aligned} \qquad (7.25)$$

$$\begin{aligned}\mathbf{OP} \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{OP})] &= \{(prxy - p^2zy - q^2zy + qry^2) - (pqxz - p^2zy - r^2zy + qrz^2)\}\mathbf{i} \\ &\quad + \{(qpyz - q^2xz - r^2xz + rpz^2) - (qryx - q^2xz - p^2xz + rpx^2)\}\mathbf{j} \\ &\quad + \{(rqzx - r^2yx - p^2yx + pqx^2) - (rpzy - r^2yx - q^2yx + pqy^2)\}\mathbf{k}\end{aligned} \qquad (7.26)$$

Let, $\mathbf{M} = M_x\,\mathbf{i} + M_y\,\mathbf{j} + M_z\,\mathbf{k}$ (7.27)

Using Eq.(7.16), Eq.(7.20) simplifies to :

$$\mathbf{M} = \int_m \mathbf{OP} \times (\dot{\boldsymbol{\omega}} \times \mathbf{OP})dm$$

$$= \int_m [(\dot{p}y^2 - \dot{q}xy - \dot{r}xz + \dot{p}z^2)\mathbf{i} + (\dot{q}z^2 - \dot{r}yz - \dot{p}yx + \dot{q}x^2)\mathbf{j} + (\dot{r}x^2 - \dot{p}zx - \dot{q}zy + \dot{r}y^2)\mathbf{k}]dm$$

$$= (A\dot{p} - F\dot{q} - E\dot{r})\mathbf{i} + (B\dot{q} - D\dot{r} - F\dot{p})\mathbf{j} + (C\dot{r} - E\dot{p} - D\dot{q})\mathbf{k} \qquad (7.28)$$

where,

$$A = I_{xx} = \int_m (y^2+z^2)dm,\ B = I_{yy} = \int_m (x^2+z^2)\,dm;$$

$$C = I_{zz} = \int_m (x^2+y^2)dm,\ D = I_{yz} = I_{zy} = \int_m y\,z\,dm; \qquad (7.29)$$

$$E = I_{zx} = I_{xz} = \int_m xz\,dm,\ F = I_{yx} = I_{xy} = \int_m x\,y\,dm;$$

$$\int_m \mathbf{OP} \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{OP})]dm = \int_m [\{(prxy - p^2zy - q^2zy + qry^2) - (pqxz - p^2zy - r^2zy + qrz^2)\}\mathbf{i}$$

$$+\{(qpyz - q^2\,xz - r^2xz + rpz^2) - (qryx - q^2\,xz - p^2xz + rpx^2)\}\mathbf{j}$$

$$+\{(rqzx - r^2\,yx - p^2yx + pqx^2) - (rpzy - r^2yx - q^2yx + pqy^2)\}\,\mathbf{k}]\,dm \qquad (7.30)$$

$$= [qr\int_m (y^2 - z^2)dm + Fpr - D(q^2 - r^2) - Epq]\mathbf{i} + [rp\int_m (z^2 - x^2)\,dm + Dqp - E(r^2 - p^2) - Fqr]\,\mathbf{j}$$

$$+[pq\int_m (x^2 - y^2)dm + Erq - F(p^2 - q^2) - Drp]\mathbf{k} \qquad (7.31)$$

Equation (7.31) is simplified by noting that:

$$\int_m (y^2 - z^2)dm = \int_m [(y^2+x^2) - (z^2+x^2)]dm = (C-B)$$

$$\int_m (z^2 - x^2)dm = \int_m [(z^2+y^2) - (x^2+y^2)]dm = (A-C) \qquad (7.32)$$

$$\int_m (x^2 - y^2)dm = \int_m [(x^2+z^2) - (y^2+z^2)]dm = (B-A)$$

Hence, $\mathbf{M} = (A\dot{p} - F\dot{q} - E\dot{r})\mathbf{i} + (B\dot{q} - D\dot{r} - F\dot{p})\mathbf{j} + (C\dot{r} - E\dot{p} - D\dot{q})\mathbf{k} + [(C - B)qr + Fpr - D(q^2 - r^2)$
$- E\,pq]\mathbf{i} + [(A - C)rp + Dqp - E(r^2 - p^2) - Fqr]\mathbf{j} + [(B - A)pq + Erq - F(p^2 - q^2) - Drp]\mathbf{k}$ (7.33)

Using Eq.(7.27) in Eq.(7.33) yields:

$$M_x = A\dot{p} - (B - C)qr + D(r^2 - q^2) - E(pq + \dot{r}) + F(pr - \dot{q}) \qquad (7.34)$$

$$M_y = B\dot{q} - (C-A)rp + E(p^2-r^2) - F(qr+\dot{p}) + D(qp-\dot{r}) \quad (7.35)$$

$$M_z = C\dot{r} - (A-B)pq + F(q^2-p^2) - D(rp+\dot{q}) + E(rq-\dot{p}) \quad (7.36)$$

7.3 Forces acting on the airplane

The external forces acting on an airplane are the thrust (**T**), the aerodynamic forces (**A**) (lift, drag and side force) and the gravitational force (m**g**). In vector form, Eq.(7.11) can be expressed as :

$$\mathbf{T} + \mathbf{A} + m\,\mathbf{g} = \int a\,dm \quad (7.37)$$

It may be recalled that the following assumptions have been made during the above derivation.

(a) The airplane is rigid.

(b) The reference frame attached to the earth is a Newtonian frame.

(c) Flat earth model is used for gravitational force.

Before obtaining the scalar form of Eq.(7.37) the following points may be noted.

(a) The thrust vector acts roughly along the fuselage reference line (FRL).

(b) The aerodynamic forces are resolved so that the drag is parallel to the free stream direction and the lift and the side force are in mutually perpendicular directions to the free stream.

(c) The gravitational force acts vertically downwards.

(d) To obtain the scalar form of Eq. (7.37), **T**, **A** and m**g** must be expressed in a single coordinate system. Towards this, different coordinate systems are defined and the relationships between them are derived in the next two sections.